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Does a Singular Symmetric Interval Matrix Contain a Symmetric Singular Matrix?

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Technical report No. V-1268

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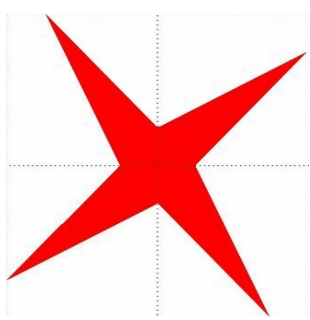
We consider the conjecture formulated in the title concerning existence of a symmetric singular matrix in a singular symmetric interval matrix. We show by means of a counterexample that it is generally not valid, and we prove that it becomes true under an additional assumption of positive semidefiniteness of the midpoint matrix. The proof is constructive.[2]



Keywords:
Symmetric interval matrix, singularity, positive semidefiniteness.

[1]This work was supported with institutional support RVO:67985807.

[2]Above: logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [?])).

Does a Singular Symmetric Interval Matrix Contain a Symmetric Singular Matrix?

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Abstract

We consider the conjecture formulated in the title concerning existence of symmetric singular matrix in a singular symmetric interval matrix. We show by means of a counterexample that it is generally not valid, and we prove that it becomes true under an additional assumption of positive semidefiniteness of the midpoint matrix. The proof is constructive.

Keywords: symmetric interval matrix, singularity, positive semidefiniteness
2010 MSC: 15A09, 65G40

1. Introduction

A square interval matrix $\boldsymbol{A} = [A - D, A + D]$ is called singular if it contains a singular matrix, and it is said to be symmetric if both A and D are symmetric. Thus unless $D = 0$, $\boldsymbol{A}$ contains nonsymmetric matrices as well. This context – namely, presence of both symmetric and nonsymmetric matrices within $\boldsymbol{A}$ – leads to a natural question: if a symmetric $\boldsymbol{A}$ is singular, does it necessarily contain a symmetric singular matrix?

In Section 2 we show by means of a 2×2 counterexample that this conjecture is not true; but then in Section 3 we prove that under an additional assumption of positive semidefiniteness of the midpoint A it becomes valid. The proof is constructive, and in Section 4 we translate it into the form of an algorithm. It is interesting that it is a two-stage process: first we must find an arbitrary (generally nonsymmetric) singular matrix in $\boldsymbol{A}$, and then we

[*] Work supported with institutional support RVO:67985807.
[**] Dedicated to Professor Ilja Černý on the occasion of his 90th birthday.
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exploit the sign structure of its null vector to construct a symmetric singular matrix in $\boldsymbol{A}$.

2. Counterexample

The symmetric interval matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & [-1, 1] \\ [-1, 1] & 1 \end{pmatrix}$$

is obviously singular since it contains the singular matrix

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$$

yet each symmetric matrix in $\boldsymbol{A}$ is of the form

$$A_t = \begin{pmatrix} -1 & t \\ t & 1 \end{pmatrix}, \quad t \in [-1, 1]$$

and it satisfies $\det(A_t) = -1 - t^2 < 0$, i.e., it is nonsingular. Hence, a singular symmetric interval matrix does *not* contain a symmetric singular matrix in the general case.

3. Existence of a symmetric singular matrix

We shall show, however, that under an additional assumption the conjecture becomes true.

Theorem 1. *A singular symmetric interval matrix $[A - D, A + D]$ with positive semidefinite A contains a symmetric singular matrix.*

Proof. By assumption there exists a singular matrix $S_0 \in [A - D, A + D]$ and thus also a vector $x \neq 0$ satisfying $S_0 x = 0$. Then we have

$$x^T A x \leq |x^T (A - S_0) x| \leq |x|^T |A - S_0| |x| \leq |x|^T D |x|. \tag{1}$$

Define a diagonal matrix T by $T_{ii} = 1$ if $x_i \geq 0$ and $T_{ii} = -1$ otherwise, then $|x| = Tx$ and substituting into (1) we obtain

$$x^T (A - TDT) x \leq 0.$$

Because $A-TDT$ is symmetric, by the Courant-Fischer theorem [1] we have

$$\lambda_{\min}(A-TDT) = \min_{x'\neq 0} \frac{x'^T(A-TDT)x'}{x'^Tx'} \leq \frac{x^T(A-TDT)x}{x^Tx} \leq 0.$$

Now define a function f of one real variable by

$$f(t) = \lambda_{\min}(A-tTDT), \quad t \in [0,\, 1].$$

Then $f(0) = \lambda_{\min}(A) \geq 0$ because A is positive semidefinite by assumption, $f(1) = \lambda_{\min}(A-TDT) \leq 0$ as proved above, and, moreover, f is continuous in $[0,\, 1]$ since by the Wielandt-Hofman theorem [1] for each $t_1, t_2 \in [0,\, 1]$ we have

$$|f(t_1) - f(t_2)| \leq \|(t_1-t_2)TDT\|_F \leq |t_1-t_2|\|D\|_F,$$

where $\|\cdot\|_F$ is the Frobenius norm. In this way the assumptions of the intermediate value theorem are met, hence there exists a $t^* \in [0,\, 1]$ such that $f(t^*) = 0$. Then

$$S = A - t^*TDT$$

is a symmetric singular matrix in $[A-D,\, A+D]$.

4. Computation of a symmetric singular matrix

We may now sum up the construction given in the proof into the form of an algorithm. Notice that first a singular matrix S_0 must be constructed (by arbitrary means; we recommend the MATLAB file mentioned in the footnote) and then the sign structure of its null vector x is exploited to construct a real function f whose zero on the interval $[0,\, 1]$ must be found (we recommend to use the classical bisection method which works well despite the lack on any additional information about f).

1. Find a singular matrix[1] $S_0 \in [A-D,\, A+D]$.
2. Find an $x \neq 0$ satisfying $S_0x = 0$.
3. $T = I$; set $T_{ii} = -1$ whenever $x_i < 0$.
4. $C = TDT$.
5. Construct a function $f(t) = \lambda_{\min}(A-tC)$, $t \in [0,\, 1]$.
6. Find a zero[2] t^* of $f(t)$ in $[0,\, 1]$.

[1] E.g. by the file available at `http://uivtx.cs.cas.cz/~rohn/other/regising.m`.

[2] E.g. by the interval halving (bisection) method.

7. $S = A - t^* C$.

Consider a randomly generated symmetric positive semidefinite integer matrix A and a symmetric nonnegative integer matrix D.

```
A =

   208    97    -8   153    62   -89
    97   197  -102    71    10   -60
    -8  -102   154   -64    -2   -17
   153    71   -64   263    54   -32
    62    10    -2    54    35   -12
   -89   -60   -17   -32   -12   186


D =

     2     4     7     2     5     2
     4     7     1     6     7     7
     7     1     2     6     8     7
     2     6     6     2     8     6
     5     7     8     8     6     5
     2     7     7     6     5     5
```

The computed matrix S_0 is not yet symmetric, but it contains a symmetric integer submatrix $A(2:5, 2:5)$. This nice integer substructure is however destroyed while computing the symmetric singular matrix S which contains no more integer entry. Finally we compute the rank of S to demonstrate its singularity.

```
S0 =

  208.5947   98.1894   -5.9186  153.5947   63.4867  -88.4053
   93.0000  190.0000 -103.0000   65.0000    3.0000  -67.0000
  -15.0000 -103.0000  152.0000  -70.0000  -10.0000  -24.0000
  151.0000   65.0000  -70.0000  261.0000   46.0000  -38.0000
   57.0000    3.0000  -10.0000   46.0000   29.0000  -17.0000
  -87.0000  -53.0000  -10.0000  -26.0000   -7.0000  191.0000
```

```
S =

  207.0808   98.8385   -4.7827  153.9192   64.2981  -89.9192
   98.8385  193.7827 -102.4596   68.2423    6.7827  -56.7827
   -4.7827 -102.4596  153.0808  -66.7577   -5.6769  -13.7827
  153.9192   68.2423  -66.7577  262.0808   50.3231  -29.2423
   64.2981    6.7827   -5.6769   50.3231   32.2423   -9.7019
  -89.9192  -56.7827  -13.7827  -29.2423   -9.7019  183.7019

>> rank(S)

ans =

     5
```

References

[1] G. H. Golub, C. F. van Loan, Matrix Computations, The Johns Hopkins University Press, Baltimore, 1996.