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Generalization of a Theorem on Eigenvalues of Symmetric Matrices

Jiří Rohn
http://uivtx.cs.cas.cz/∼rohn

Technical report No. V-1271

22.07.2019

Pod Vodárenskou věží 2, 182 07 Prague 8, phone: +420 266 051 111, fax: +420 286 585 789,
e-mail:rohn@cs.cas.cz

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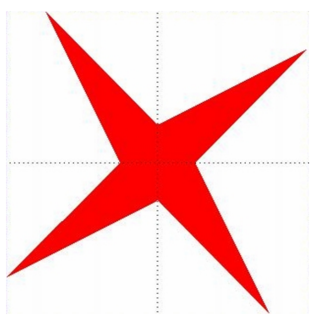
Abstract:

We prove that the product of a symmetric positive semidefinite matrix and a symmetric matrix has all eigenvalues real. [2]



Keywords:
Symmetric matrix, positive semidefinite matrix, real spectrum.

[1]This work was supported with institutional support RVO:67985807.

[2]Above: Logo of interval computations and related areas (depiction of the solution set of the system $[2,4]x_1 + [-2,1]x_2 = [-2,2]$, $[-1,2]x_1 + [2,4]x_2 = [-2,2]$ (Barth and Nuding [1])).

A real matrix has complex eigenvalues in general. Yet there is a well-known important exception:

Theorem 1. *A symmetric matrix $A \in \mathbb{R}^{n\times n}$ has all eigenvalues real.*

In this note we prove a generalization of this result and we show a way how to construct generally nonsymmetric matrices having real eigenvalues only.

Theorem 2. *If $A, B \in \mathbb{R}^{n\times n}$ are symmetric matrices of which at least one is positive semidefinite, then AB has all eigenvalues real.*

Proof. (a) First, let A be positive semidefinite. Put $C = A^{1/2}$, so that C is the unique symmetric positive semidefinite real matrix satisfying $C^2 = A$ (Horn and Johnson [2, Thm. 7.2.6]). Then

$$AB = C^2 B = CC^T B = C(C^T B)$$

and since $C(C^T B)$ and $(C^T B)C$ have the same spectrum (Horn and Johnson [2, Thm. 1.3.20]), the eigenvalues of AB and $(C^T B)C$ are equal, hence it suffices to prove that $C^T BC$ has all eigenvalues real. Thus let $\lambda \in \mathbb{C}$ be an eigenvalue of $C^T BC$, i.e.,

$$C^T BCx = \lambda x \tag{0.1}$$

holds for some $0 \neq x \in \mathbb{C}^n$ which can be normalized so that $x^* x = 1$, where x^* denotes the conjugate transpose. Premultiplying (0.1) by x^* yields

$$\lambda = x^* C^T BCx = y^* By = \sum_{ij} B_{ij} y_i^* y_j,$$

where $y = Cx \in \mathbb{C}^n$, and consequently

$$\lambda = \sum_i B_{ii}|y_i|^2 + \sum_{i<j}(B_{ij} y_i^* y_j + B_{ji} y_j^* y_i) = \sum_i B_{ii}|y_i|^2 + \sum_{i<j} B_{ij}(y_i^* y_j + y_j^* y_i). \tag{0.2}$$

Because

$$(y_i^* y_j + y_j^* y_i)^* = y_i^* y_j + y_j^* y_i,$$

the number $y_i^* y_j + y_j^* y_i$, being equal to its complex conjugate, is real and this in the light of (0.2) means that λ is also real.

(b) If B is positive semidefinite then by the part (a) above BA has all eigenvalues real and thus the same holds for AB whose spectrum equals to that of BA. □

To show that Theorem 2 is indeed a generalization of Theorem 1, let us decompose a given symmetric matrix A as $A = AI$ where I is the identity matrix, then the assumption of Theorem 2 are met which implies that all eigenvalues of A are real.

Finally we describe a way how to generate generally nonsymmetric matrices having real eigenvalues only.

Theorem 3. *For any $A, B \in \mathbb{R}^{n\times n}$, the matrices*

$$A^T A B^T B$$

and

$$A^T A(B + B^T)$$

have all eigenvalues real.

Proof. This follows immediately from Theorem 2 since $A^T A$ is symmetric positive semidefinite, and $B^T B$ and $B + B^T$ are symmetric. □

Bibliography

[1] W. Barth and E. Nuding, *Optimale Lösung von Intervallgleichungssystemen*, Computing, 12 (1974), pp. 117–125.

[2] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.