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A Residual Existence Theorem for Linear Equations

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Abstract A residual existence theorem for linear equations is proved: if $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and if X is a finite subset of $\mathbb{R}^n$ satisfying $\max_{x\in X} p^T(Ax - b) \geq 0$ for each $p \in \mathbb{R}^m$, then the system of linear equations $Ax = b$ has a solution in the convex hull of X. An application of this result to unique solvability of the absolute value equation $Ax + B|x| = b$ is given.

Keywords Linear equations · Solution · Existence · Residual · Convex hull · Absolute value equation

1 Introduction

As the main result of this paper, we prove the following residual existence (and localization) theorem for linear equations (Theorem 2 below): if $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ and if X is a finite subset of $\mathbb{R}^n$ satisfying

$$\max_{x\in X} p^T(Ax - b) \geq 0$$

for each $p \in \mathbb{R}^m$, then the system of linear equations

$$Ax = b$$

has a solution in the convex hull of X. The result is then applied to derive a sufficient condition for unique solvability of the absolute value equation

$$Ax + B|x| = b$$

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(Theorem 6).

We use the following notations. Matrix (or vector) inequalities, as $A \leq B$ or $A < B$, are understood componentwise. $A_{\bullet i}$ denotes the ith column of A. I is the unit matrix and $e = (1, \ldots, 1)^T$ is the vector of all ones. The absolute value of a matrix (or vector) $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. $Y_n = \{y \mid |y| = e\}$ is the set of all ± 1-vectors in $\mathbb{R}^n$, so that its cardinality is 2^n. For each $y \in \mathbb{R}^n$ we denote

$$T_y = \operatorname{diag}(y_1, \ldots, y_n) = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & y_n \end{pmatrix}, \tag{1}$$

and $\operatorname{Conv}(X)$ is the convex hull of X.

2 The residual existence theorem

In the proof of the main theorem we shall utilize the following result proved by Gordan [2] (see also [1]).

Theorem 1 *Let $A \in \mathbb{R}^{m \times n}$. Then the system*

$$\begin{aligned} Ax &= 0, \\ e^T x &= 1, \\ x &\geq 0 \end{aligned}$$

has a solution if and only if for each $p \in \mathbb{R}^m$ there holds

$$(A^T p)_i \leq 0$$

for some i.

The following theorem is the principal result of this paper.

Theorem 2 *Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and let X be a finite subset of $\mathbb{R}^n$ such that*

$$\max_{x \in X} p^T (Ax - b) \geq 0 \tag{2}$$

holds for each $p \in \mathbb{R}^m$. Then the system

$$Ax = b$$

has a solution in $\operatorname{Conv}(X)$.

Comment We call this result a "residual existence theorem" because the condition (2) is formulated in terms of a finite set of residuals $Ax - b$, $x \in X$.

Proof Let $X = \{x_1, \ldots, x_k\}$, and let R be an $m \times k$ matrix defined by $R_{\bullet i} = b - Ax_i$ ($i = 1, \ldots, k$). By (2), for each $p \in \mathbb{R}^m$ there exists an i such that $p^T(Ax_i - b) \geq 0$, hence

$$(R^T p)_i = (p^T R)_i = p^T(b - Ax_i) = -p^T(Ax_i - b) \leq 0$$

holds, thus by Theorem 1 there exists a vector $\lambda \in \mathbb{R}^k$ satisfying

$$R\lambda = 0, \tag{3}$$
$$e^T\lambda = 1, \tag{4}$$
$$\lambda \geq 0. \tag{5}$$

Then (3) gives

$$\sum_{i=1}^{k} \lambda_i(b - Ax_i) = 0, \tag{6}$$

so that the vector

$$x = \sum_{i=1}^{k} \lambda_i x_i$$

in view of (4), (5), (6) satisfies

$$Ax = (e^T\lambda)b = b$$

and

$$x \in \mathrm{Conv}(X),$$

which concludes the proof. □

The condition (2) is generally not easy to verify, but it is satisfied if the residual set

$$\{\, Ax - b \mid x \in \mathbb{R}^n \,\} \tag{7}$$

intersects all orthants of $\mathbb{R}^m$. In this way we obtain the following consequence.

Theorem 3 *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then the system of linear equations*

$$Ax = b \tag{8}$$

has a solution if and only if the residual set (7) intersects all orthants of $\mathbb{R}^m$.

Proof If (8) has a solution x, then $b - Ax = 0$ belongs to each orthant of $\mathbb{R}^m$. Conversely, if (7) intersects all orthants of $\mathbb{R}^m$, then for each orthant $\mathcal{O}$ we can pick an $x_{\mathcal{O}}$ satisfying $Ax_{\mathcal{O}} - b \in \mathcal{O}$. Put $X = \{\, x_{\mathcal{O}} \mid \mathcal{O}$ is an orthant of $\mathbb{R}^m \,\}$. Then for each $p \in \mathbb{R}^m$, letting $\mathcal{O}$ to be the orthant containing p, we have $p^T(Ax_{\mathcal{O}} - b) \geq 0$ and Theorem 2 implies existence of a solution to (8). □

A small change in the definition of the residual set (7) makes it possible to formulate an analogous result for nonnegative solvability.

Theorem 4 *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then the system of linear equations*

$$Ax = b \tag{9}$$

has a nonnegative solution if and only if the residual set

$$\{\, Ax - b \mid x \geq 0 \,\} \tag{10}$$

intersects all orthants of $\mathbb{R}^m$.

Proof Obviously, if (9) has a nonnegative solution, then the set (10) contains 0 and thus intersects all orthants. Conversely, if the latter is true, then for each orthant $\mathcal{O}$ of $\mathbb{R}^m$ there exists a *nonnegative* $x_{\mathcal{O}}$ satisfying $Ax_{\mathcal{O}} - b \in \mathcal{O}$, and arguing as in the proof of Theorem 3 we come to the conclusion that the equation (9) has a solution which belongs to the convex hull of the nonnegative vectors $x_{\mathcal{O}}$ and thus is itself nonnegative as well. ☐

3 Application: Unique solvability of the absolute value equation

As an application of our previous results, we prove an existence and uniqueness theorem for the absolute value equation

$$Ax + B|x| = b$$

(A, B square) which has been recently studied in literature (Mangasarian [3], [4], [5], Mangasarian and Meyer [6], Prokopyev [7], Rohn [8]). The basic result concerning unique solvability of the absolute value equation is Theorem 4.1 in [8].

Theorem 5 *For each $A, B \in \mathbb{R}^{n\times n}$, exactly one of the two alternatives holds:*

(i) for each B' with $|B'| \le |B|$ and for each $b \in \mathbb{R}^n$ the equation

$$Ax + B'|x| = b$$

has a unique solution,

(ii) the inequality

$$|Ax| \le |B||x|$$

has a nontrivial solution.

We shall use this theorem to prove the following result.

Theorem 6 *Let $A, B \in \mathbb{R}^{n\times n}$ and let for each $y \in Y_n$ the equation*

$$Ax - T_y|B||x| = y \tag{11}$$

have a solution. Then for each B' with $|B'| \le |B|$ and for each $b \in \mathbb{R}^n$ the equation

$$Ax + B'|x| = b \tag{12}$$

has a unique solution.

Comment Thus, solvability of a finite number of equations (11) (albeit 2^n of them) guarantees *unique* solvability of an infinite number of equations of the form (12) (see (1) for the definition of T_y).

Proof For each $y \in Y_n$, let x_y be a solution of (11). The main part of the proof consists in proving that each matrix C satisfying

$$|C - A| \le |B| \tag{13}$$

is nonsingular. Thus let C satisfy (13). Then for each $y \in Y_n$ we have

$$|T_y(Cx_y - Ax_y)| = |Cx_y - Ax_y| \le |C - A||x_y| \le |B||x_y|,$$

hence

$$T_y A x_y - |B||x_y| \leq T_y C x_y \leq T_y A x_y + |B||x_y|$$

and

$$\begin{aligned} T_y(Cx_y - I_{\bullet j}) &\geq T_y A x_y - |B||x_y| - T_y I_{\bullet j} = T_y(Ax_y - T_y|B||x_y| - y) + T_y y - T_y I_{\bullet j} \\ &= e - y_j I_{\bullet j} \geq 0 \end{aligned}$$

for each $j = 1, \ldots, n$ (because x_y solves (11) and $y \in Y_n$). Thus for each $j = 1, \ldots, n$ the set $\{ Cx_y - I_{\bullet j} \mid y \in Y_n \}$ intersects all the orthants, which in the light of Theorem 3 means that the system $Cx = I_{\bullet j}$ has a solution $x^{(j)}$. Define a matrix X by $X_{\bullet j} = x^{(j)}$ for $j = 1, \ldots, n$, then

$$CX = I,$$

which proves that C is nonsingular. In this way we have proved that each matrix C satisfying (11) is nonsingular, which shows that the inequality $|Ax| \leq |B||x|$ has only the trivial solution $x = 0$ ([8], Proposition 2.2). Now Theorem 5 guarantees unique solvability of the equation (12) for each B' with $|B'| \leq |B|$ and for each right-hand side b. □

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