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On Rump's Characterization of P-Matrices

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Abstract The necessary and sufficient P-matrix condition by S. M. Rump is simplified by showing that one of its assumptions can be deleted without affecting validity of the result.

Keywords P-matrix · interval matrix · regularity.

1 Introduction

S. M. Rump in his paper [3] proved the following result concerning a square matrix A.

Theorem 1 *Let both $A - I$ and $A + I$ be nonsingular. Then A is a P-matrix if and only if the interval matrix*

$$[(A - I)^{-1}(A + I) - I, (A - I)^{-1}(A + I) + I] \tag{1}$$

is regular.

Here, I is the identity matrix, an interval matrix is a set of matrices of the form $[B, C] = \{ X \mid B \le X \le C \}$ with entrywise comparison, and an interval matrix is called regular if each point matrix contained therein is nonsingular. The main purpose of the present paper consists in showing that the assumption of nonsingularity of $A+I$ is redundant and may be deleted without affecting validity of the theorem. The main result (Theorem 2) is preceded by two auxiliary propositions that make it easier to understand the relationship between the P-property of real matrices and regularity of interval matrices. See also [1] for related results.

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2 Definitions and notation

Given an $n \times n$ matrix A and a subset $\emptyset \neq J \subseteq \{1, \ldots, n\}$, denote by $A[J]$ the submatrix of A consisting of rows and columns whose indices belong to J. Submatrices formed in this way are called principal submatrices, and A is said to be a P-matrix if determinants of all the principal submatrices (also called principal minors) are positive.

We have defined, as usual, $A[J]$ for $J \neq \emptyset$ only, but we shall also need to have it defined for $J = \emptyset$. In this case we define $A[\emptyset]$ to be the empty matrix, and we set $\det(A[\emptyset]) = 1$.

For each vector $y \in Y_n = \{-1, 1\}^n$ (the set of all ± 1-vectors in $\mathbb{R}^n$) we put

$$A_y = (A - I)^{-1}(A + I) - T_y, \tag{2}$$

where $T_y = \mathrm{diag}(y)$ is the diagonal matrix with diagonal vector y. Obviously,

$$A_y \in [(A - I)^{-1}(A + I) - I, (A - I)^{-1}(A + I) + I]$$

for each such a y.

3 The result

First we show that regularity of the interval matrix (1) can be formulated in terms of determinants of the matrices A_y.

Proposition 1 *Let $A - I$ be nonsingular. Then the interval matrix*

$$[(A - I)^{-1}(A + I) - I, (A - I)^{-1}(A + I) + I]$$

is regular if and only if the numbers

$$\det(A_y), \quad y \in Y_n$$

are either all simultaneously positive, or all simultaneously negative.

Proof In [2, Thm. 5.1, (C1)] it is proved that an interval matrix $[A_c - \Delta, A_c + \Delta]$ is regular if and only if the determinants of the matrices

$$A_c - T_{y'} \Delta T_{z'}, \quad y', z' \in Y_n$$

are either all simultaneously positive, or all simultaneously negative. In our case we have

$$A_c - T_{y'} \Delta T_{z'} = (A - I)^{-1}(A + I) - T_y = A_y$$

where y is given by $y_i = y'_i z'_i$ $(i = 1, \ldots, n)$, hence the general condition reduces to the respective property of matrices A_y, $y \in Y_n$. □

Next we show a connection between determinants of matrices A_y and principal minors of A.

Proposition 2 *Let $A - I$ be nonsingular. Then for each $y \in Y_n$ we have*

$$\det(A_y) = \frac{2^n \det(A[J(y)])}{\det(A - I)}, \tag{3}$$

where

$$J(y) = \{ j \mid y_j = -1 \}.$$

Proof Let $y \in Y_n$. Then from (2) we have

$$(A - I)A_y = A + I - (A - I)T_y = A(I - T_y) + I + T_y \tag{4}$$

and for the jth column of the right-hand side matrix there holds

$$(A(I - T_y) + I + T_y)e_j = \begin{cases} 2e_j & \text{if } y_j = 1, \\ 2Ae_j & \text{if } y_j = -1, \end{cases}$$

where e_j denotes the jth column of I ($j = 1, \ldots, n$). Taking the Laplace expansion along all the columns with $y_j = 1$, we obtain

$$\det(A(I - T_y) + I + T_y) = 2^n \det(A[J(y)]) \tag{5}$$

and this result also holds for $J(y) = \emptyset$ because in this case both sides in (5) are equal to 2^n in view of our definition of $\det(A[\emptyset])$ in Section 2. Hence, from (4) and (5) we obtain

$$\det(A - I)\det(A_y) = 2^n \det(A[J(y)])$$

and

$$\det(A_y) = \frac{2^n \det(A[J(y)])}{\det(A - I)},$$

which concludes the proof. □

Finally, we prove our main result.

Theorem 2 *Let $A - I$ be nonsingular. Then A is a P-matrix if and only if the interval matrix*

$$[(A - I)^{-1}(A + I) - I, (A - I)^{-1}(A + I) + I] \tag{6}$$

is regular.

Proof If A is a P-matrix, then $\det(A[J(y)]) > 0$ for each $y \in Y_n$ and by (3) the determinants of all the matrices A_y, $y \in Y_n$ are of the same sign (positive if $\det(A - I) > 0$ and negative if $\det(A - I) < 0$), hence the interval matrix (6) is regular due to Proposition 1. Conversely, if the interval matrix (6) is regular, then determinants of all the matrices A_y are of the same sign by Proposition 1 which in view of (3) means that all the numbers $\det(A[J(y)])$, $y \in Y_n$ are of the same sign. But since for $y = (1, \ldots, 1)^T$ we have $\det(A[J(y)]) = \det(A[\emptyset]) = 1$, all the principal minors are positive and A is a P-matrix. □

This proves our original claim.

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References

1. Johnson, C.R., Tsatsomeros, M.J.: Convex sets of nonsingular and P-matrices. Linear and Multilinear Algebra **38**, 233–239 (1995)
2. Rohn, J.: Systems of linear interval equations. Linear Algebra and Its Applications **126**, 39–78 (1989). DOI 10.1016/0024-3795(89)90004-9
3. Rump, S.M.: On P-Matrices. Linear Algebra and Its Applications **363**, 237–250 (2003). DOI 10.1016/S0024-3795(01)00590-0