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SOLVABILITY OF SYSTEMS OF LINEAR INTERVAL EQUATIONS*

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Abstract. A system of linear interval equations is called solvable if each system of linear equations contained therein is solvable. In the main result of this paper it is proved that solvability of a general rectangular system of linear interval equations can be characterized in terms of nonnegative solvability of a finite number of systems of linear equations which, however, is exponential in matrix size; the problem is proved to be NP-hard. It is shown that three earlier published results are consequences of the main theorem, which is compared with its counterpart valid for linear interval inequalities that turn out to be much less difficult to solve.

Key words. linear interval equations, solvability, complexity, linear interval inequalities

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1. Introduction.

Let $\mathbf{A} = [\underline{A}, \overline{A}] = \{A;\ \underline{A} \leq A \leq \overline{A}\}$ be an $m \times n$ interval matrix and $\mathbf{b} = [\underline{b}, \overline{b}] = \{b;\ \underline{b} \leq b \leq \overline{b}\}$ an m-dimensional interval vector (inequalities are taken componentwise and it is assumed that $\underline{A} \leq \overline{A}$ and $\underline{b} \leq \overline{b}$, so that both sets are nonempty). A system of linear interval equations, formally written as

$$\mathbf{A}x = \mathbf{b}, \tag{1}$$

is defined to be the family of all systems of linear equations

$$Ax = b \tag{2}$$

with data satisfying

$$A \in \mathbf{A}, \qquad b \in \mathbf{b}. \tag{3}$$

During approximately the last 35 years, much attention has been paid to systems of linear interval equations (1) with square interval matrices (cf., e.g., the monographs by Alefeld and Herzberger [1], Neumaier [4], Kreinovich et al. [3]). On the contrary, the general rectangular case has been much less studied and remains much less understood.

In this paper we raise the question of solvability of general systems of linear interval equations with rectangular matrices. A system (1) is called solvable if each system in the family (2), (3) is solvable (i.e., has a solution). The reasons for introducing this property are obvious: assuming we are interested in solvability of a linear system $A_0x = b_0$, whose data A_0, b_0 are not known exactly but only known to belong to $\mathbf{A}$ and $\mathbf{b}$, respectively, we can be sure that the system $A_0x = b_0$ is solvable only if each system (2) with data satisfying (3) possesses this property.

Except for the trivial case of $\underline{A} = \overline{A}$ and $\underline{b} = \overline{b}$, the family (2), (3) consists of infinitely many linear systems. In the main result of this paper (Theorem 3) we prove that a system (1) is solvable if and only if a finite number of linear systems are

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nonnegatively solvable (i.e., have nonnegative solutions). These systems are formed in the following way: For each $i \in \{1, \ldots, m\}$, the ith equation of such a system is either of the form

$$(\underline{A}x^1 - \overline{A}x^2)_i = \overline{b}_i \tag{4}$$

or of the form

$$(\overline{A}x^1 - \underline{A}x^2)_i = \underline{b}_i. \tag{5}$$

Since for each of the m equations we have two options to choose from, there are altogether 2^m linear systems of this form in general (notice that the matrix of each such system is of size $m \times 2n$). But if the ith rows of $\underline{A}$ and $\overline{A}$ are equal and if $\underline{b}_i = \overline{b}_i$, then (4) and (5) coincide. Hence the exact number of mutually different linear systems to be solved is 2^q, where q is the number of nonzero rows of the matrix $(\overline{A} - \underline{A}, \overline{b} - \underline{b})$. This shows that the characterization, although generally exponential, can be of practical use for problems with moderate values of q.

As shown in section 3, the proof of this result is nontrivial and relies on the Farkas lemma and on the Oettli–Prager theorem. In section 4 we show that the main result offers a unified view of three different, earlier results published independently: characterization of nonnegative solvability of (1) (Theorem 4), characterization of regularity of interval matrices (Theorem 5), and the convex-hull theorem (Theorem 6). Next it is shown that the problem of checking solvability of linear interval equations is NP-hard (Theorem 7); this explains the exponentiality inherent in formulation of the main result. Finally, we compare the characterization of solvability of linear interval equations in Theorem 3 with that of linear interval inequalities. Unlike the case of exact data, these two problems turn out to be of different complexity since solvability of a system of linear interval inequalities is characterized by solvability of *one* system of linear inequalities only (Theorem 8). A brief discussion of the reasons for this difference concludes the paper.

Throughout the paper we shall use the following notation. For an interval matrix $\mathbf{A} = [\underline{A}, \overline{A}]$ we define

$$A_c = \tfrac{1}{2}(\underline{A} + \overline{A})$$

(the center matrix) and

$$\Delta = \tfrac{1}{2}(\overline{A} - \underline{A})$$

(the radius matrix). Then $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$, so that we also can write $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$. Similarly, for the right-hand side $\mathbf{b} = [\underline{b}, \overline{b}]$, setting

$$b_c = \tfrac{1}{2}(\underline{b} + \overline{b})$$

and

$$\delta = \tfrac{1}{2}(\overline{b} - \underline{b}),$$

we have $\mathbf{b} = [b_c - \delta, b_c + \delta]$. This form of expressing the bounds turns out to be more useful, mainly due to the Oettli–Prager description of the solution set of (1) (Theorem 2 below). For a vector $x = (x_i)$, its absolute value is defined by $|x| = (|x_i|)$; Conv X denotes the convex hull of X. We define

$$Y_m = \{y \in \mathbb{R}^m;\ y_j \in \{-1, 1\} \text{ for each } j\};$$

i.e., Y_m is the set of all ± 1-vectors in $\mathbb{R}^m$; its cardinality is obviously 2^m. Finally, for each $y \in Y_m$ we denote

$$T_y = \operatorname{diag}(y_1, \ldots, y_m) = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & y_m \end{pmatrix}.$$

Notice that $A_c - T_y\Delta \in \mathbf{A}$, $A_c + T_y\Delta \in \mathbf{A}$, and $b_c + T_y\delta \in \mathbf{b}$ for each $y \in Y_m$ (these quantities appear in formulation of the main result, equation (8) below).

2. Preliminaries. In order to keep the paper self-contained, we give here explicit formulations of two well-known results that will be used in the proof of the main theorem. The first is the Farkas lemma.

LEMMA 1 (Farkas [2]). *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then the system*

$$\begin{aligned} Ax &= b, \\ x &\geq 0, \end{aligned}$$

has a solution if and only if each $p \in \mathbb{R}^m$ with $A^T p \geq 0$ satisfies $b^T p \geq 0$.

Our second auxiliary result is the Oettli–Prager theorem. If $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is an $m \times n$ interval matrix and $\mathbf{b} = [b_c - \delta, b_c + \delta]$ is an m-dimensional interval vector, then the solution set of the system of linear interval equations

$$\mathbf{A}x = \mathbf{b}$$

is defined by

$$X = \{x;\ Ax = b \text{ for some } A \in \mathbf{A}, b \in \mathbf{b}\}. \tag{6}$$

The Oettli–Prager theorem gives a description of the solution set by means of a certain nonlinear inequality.

THEOREM 2 (Oettli and Prager [5]). *We have*

$$X = \{x;\ |A_c x - b_c| \leq \Delta|x| + \delta\}. \tag{7}$$

Hence, if x satisfies the inequality in (7), then $Ax = b$ for some $A \in \mathbf{A}$ and $b \in \mathbf{b}$. In fact, A and b can be explicitly expressed in terms of x (see the proof of Theorem 2.1 in [8]), but we shall not need it in this paper.

3. Solvability. In this section we present the main result of this paper, a characterization of solvability of linear interval equations defined in the following way. Let $\mathbf{A}$ be an $m \times n$ interval matrix and $\mathbf{b}$ an m-dimensional interval vector. The system of linear interval equations $\mathbf{A}x = \mathbf{b}$ is said to be *solvable* if each system $Ax = b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ has a solution.

Except for the trivial case $\Delta = 0$ and $\delta = 0$, the family $\mathbf{A}x = \mathbf{b}$ consists of infinitely many systems. Yet the following theorem shows that solvability of $\mathbf{A}x = \mathbf{b}$ can be characterized in terms of nonnegative solvability of a finite number of linear systems, although this number is generally exponential in matrix size.

Theorem 3. *A system of linear interval equations* $\mathbf{A}x = \mathbf{b}$ *is solvable if and only if for each* $y \in Y_m$ *the system*

$$(A_c - T_y\Delta)x^1 - (A_c + T_y\Delta)x^2 = b_c + T_y\delta, \tag{8}$$

$$x^1 \geq 0, \qquad x^2 \geq 0, \tag{9}$$

has a solution x_y^1, x_y^2. *Moreover, if this is the case, then for each* $A \in \mathbf{A}$, $b \in \mathbf{b}$ *the system* $Ax = b$ *has a solution in the set*

$$\operatorname{Conv}\{x_y^1 - x_y^2;\ y \in Y_m\}.$$

Proof. "Only if": Let $\mathbf{A}x = \mathbf{b}$ be solvable. Assume to the contrary that (8), (9) does not have a solution for some $y \in Y_m$. Then the Farkas lemma implies existence of a $p \in \mathbb{R}^m$ satisfying

$$(A_c - T_y\Delta)^T p \geq 0, \tag{10}$$

$$(A_c + T_y\Delta)^T p \leq 0, \tag{11}$$

$$(b_c + T_y\delta)^T p < 0. \tag{12}$$

Now (10) and (11) together give

$$\Delta^T T_y p \leq A_c^T p \leq -\Delta^T T_y p,$$

hence

$$|A_c^T p| \leq -\Delta^T T_y p = |-\Delta^T T_y p| \leq \Delta^T |p|,$$

and the Oettli–Prager theorem as applied to the system $[A_c^T - \Delta^T, A_c^T + \Delta^T]z = [0, 0]$ shows that there exists a matrix $A \in \mathbf{A}$ such that

$$A^T p = 0. \tag{13}$$

In light of the Farkas lemma, (13) and (12) mean that the system

$$Ax = b_c + T_y\delta$$

has no solution, which contradicts our assumption since $A \in \mathbf{A}$ and $b_c + T_y\delta \in \mathbf{b}$.

"If": Conversely, let for each $y \in Y_m$ the system (8), (9) have a solution x_y^1, x_y^2. Let $A \in \mathbf{A}$, $b \in \mathbf{b}$. To prove that the system $Ax = b$ has a solution, we first show that $T_y A x_y \geq T_y b$ holds for each $y \in Y_m$, where $x_y = x_y^1 - x_y^2$. Thus let $y \in Y_m$. Then we have

$$\begin{aligned} T_y(Ax_y - b) &= T_y(A_c x_y - b_c) + T_y(A - A_c)x_y + T_y(b_c - b) \\ &\geq T_y(A_c x_y - b_c) - \Delta|x_y| - \delta \end{aligned}$$

since $|T_y(A - A_c)x_y| \leq \Delta|x_y|$, which implies $T_y(A - A_c)x_y \geq -\Delta|x_y|$, and similarly $|T_y(b_c - b)| \leq \delta$ implies $T_y(b_c - b) \geq -\delta$; thus

$$\begin{aligned} T_y(Ax_y - b) &\geq T_y(A_c(x_y^1 - x_y^2) - b_c) - \Delta|x_y^1 - x_y^2| - \delta \\ &\geq T_y(A_c(x_y^1 - x_y^2) - b_c) - \Delta(x_y^1 + x_y^2) - \delta \\ &= T_y((A_c - T_y\Delta)x_y^1 - (A_c + T_y\Delta)x_y^2 - (b_c + T_y\delta)) \\ &= 0 \end{aligned}$$

since x_y^1, x_y^2 solve (8), (9). In this way we have proved that for each $y \in Y_m$, x_y satisfies

$$T_y A x_y \geq T_y b. \tag{14}$$

Using (14), we shall next prove that the system of linear equations

$$\sum_{y \in Y_m} \lambda_y A x_y = b, \tag{15}$$

$$\sum_{y \in Y_m} \lambda_y = 1, \tag{16}$$

has a solution $\lambda_y \geq 0$, $y \in Y_m$. In view of the Farkas lemma, it suffices to show that for each $p \in \mathbb{R}^m$ and each $p_0 \in \mathbb{R}^1$,

$$p^T A x_y + p_0 \geq 0 \text{ for each } y \in Y_m \tag{17}$$

implies

$$p^T b + p_0 \geq 0. \tag{18}$$

Thus let p and p_0 satisfy (17). Define $y \in Y_m$ by $y_i = -1$ if $p_i \geq 0$ and by $y_i = 1$ if $p_i < 0$ $(i = 1, \ldots, m)$, then $p = -T_y|p|$, and from (14), (17) we have

$$p^T b + p_0 = -|p|^T T_y b + p_0 \geq -|p|^T T_y A x_y + p_0 = p^T A x_y + p_0 \geq 0,$$

which proves (18). Hence the system (15), (16) has a solution $\lambda_y \geq 0$, $y \in Y_m$. Put $x = \sum_{y \in Y_m} \lambda_y x_y$, then $Ax = b$ by (15), and x belongs to the set $\text{Conv}\{x_y;\, y \in Y_m\} = \text{Conv}\{x_y^1 - x_y^2;\, y \in Y_m\}$ by (16). This proves the "if" part and also the additional assertion. ∎

Let us have a closer look at the form of systems (8). If $y_i = 1$, then the ith rows of $A_c - T_y\Delta$ and $A_c + T_y\Delta$ are equal to the ith rows of $\underline{A}$ and $\overline{A}$, respectively, and $(b_c + T_y\delta)_i = \overline{b}_i$. This means that in this case the ith equation of (8) has the form

$$(\underline{A}x^1 - \overline{A}x^2)_i = \overline{b}_i, \tag{19}$$

and similarly, in case $y_i = -1$ it is of the form

$$(\overline{A}x^1 - \underline{A}x^2)_i = \underline{b}_i. \tag{20}$$

Hence we can see that the family of systems (8) for all $y \in Y_m$ is just the family of all systems whose ith equations are either of the form (19) or of the form (20) for $i = 1, \ldots, m$. The number of mutually different such systems is exactly 2^q, where q is the number of nonzero rows of the matrix (Δ, δ). Hence, despite the inherent exponentiality, Theorem 3 can be of practical use if q is of moderate size.

In the "if" part of the proof we proved that for each $A \in \mathbf{A}$ and $b \in \mathbf{b}$ the equation $Ax = b$ has a solution in the set $\text{Conv}\{x_y^1 - x_y^2;\, y \in Y_m\}$. The proof, relying on the Farkas lemma, was purely existential. It is worth noting, however, that such a solution can be found in a constructive way when using an algorithm described in [9]. For its description we need a special order of elements of Y_m defined inductively via the sets Y_j, $j = 1, \ldots, m-1$, in the following way:

(i) The order of Y_1 is $-1, 1$.

(ii) If $y_1, \dots, y_{2^j}$ is the order of Y_j, then $(y_1, -1), \dots, (y_{2^j}, -1), (y_1, 1), \dots, (y_{2^j}, 1)$ is the order of Y_{j+1}.

Further, for a sequence $z_1, \dots, z_{2h}$ with an even number of elements, each pair z_j, z_{j+h} is called a conjugate pair, $j = 1, \dots, h$. As in Theorem 3, for each $y \in Y_m$, let x_y^1 and x_y^2 be a solution to (8), (9). Then the algorithm runs as follows:

1. Select $A \in \mathbf{A}$ and $b \in \mathbf{b}$.
2. Form a sequence of vectors $((x_{-y}^1 - x_{-y}^2)^T, (A(x_{-y}^1 - x_{-y}^2) - b)^T)^T$ set in the order of the y's in Y_m.
3. For each conjugate pair x, x' in the current sequence compute

$$\lambda = \begin{cases} \frac{x'_k}{x'_k - x_k} & \text{if } x'_k \neq x_k, \\ 1 & \text{otherwise,} \end{cases}$$

where k is the index of the current last entry, and set

$$x := \lambda x + (1 - \lambda) x'.$$

4. Cancel the second part of the sequence and in the remaining part delete the last entry of each vector.
5. If there remains a single vector x, terminate: x solves $Ax = b$ and $x \in \operatorname{Conv}\{x_y^1 - x_y^2;\, y \in Y_m\}$. Otherwise go to step 3.

The algorithm starts with 2^m vectors $((x_{-y}^1 - x_{-y}^2)^T, (A(x_{-y}^1 - x_{-y}^2) - b)^T)^T \in \mathbb{R}^{n+m}$, $y \in Y_m$, and proceeds by halving the sequence and deleting the last entry; hence it is finite and at the end produces a single vector $x \in \mathbb{R}^n$. The assertion made in step 5 is a consequence of Theorem 2 in [9] because we have

$$T_y A x_y \geq T_y b$$

for each $y \in Y_m$; hence also

$$T_y A x_{-y} \leq T_y b$$

for each $y \in Y_m$, which is the form used in [9].

4. Remarks. In this section we show that Theorem 3 offers a unified view of three earlier published results whose original proofs were rather involved and that can be easily obtained, and perhaps also better understood, as consequences of the main result. Next we compare the results for linear interval equations with those for linear interval inequalities that, unlike the case of exact data, turn out to be of different complexity.

First we consider nonnegative solvability. A linear interval system $\mathbf{A}x = \mathbf{b}$ is called *nonnegatively solvable* if each system $Ax = b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ is nonnegatively solvable. The following characterization (without the convex hull part) was proved in [7].

THEOREM 4. *A system of linear interval equations $\mathbf{A}x = \mathbf{b}$ is nonnegatively solvable if and only if for each $y \in Y_m$ the system*

$$(A_c - T_y \Delta) x = b_c + T_y \delta \tag{21}$$

has a nonnegative solution x_y. Moreover, if this is the case, then for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ the system $Ax = b$ has a solution in the set

$$\operatorname{Conv}\{x_y;\, y \in Y_m\}.$$

Repeating the argument following the proof of Theorem 3, we can say that the ith row of (21) is of the form

$$(\underline{A}x)_i = \overline{b}_i$$

if $y_i = 1$ and of the form

$$(\overline{A}x)_i = \underline{b}_i$$

if $y_i = -1$ (hence, unlike (8), the system matrix always belongs to $\mathbf{A}$ in this case), and the number of mutually different systems (21) is again 2^q, where q is the number of nonzero rows of the matrix (Δ, δ).

Next we turn to square matrices. A square interval matrix $\mathbf{A}$ is said to be *regular* if each $A \in \mathbf{A}$ is nonsingular. A number of necessary and sufficient regularity conditions was given in Theorem 5.1 in [8]. One of them is the following, which is again obtained as an easy consequence of Theorem 3.

THEOREM 5. *An interval matrix* $\mathbf{A}$ *is regular if and only if for each* $y \in Y_m$ *the system*

$$(A_c - T_y\Delta)x^1 - (A_c + T_y\Delta)x^2 = y,$$

$$x^1 \geq 0, \qquad x^2 \geq 0,$$

has a solution.

If $\mathbf{A}$ is regular, then for each right-hand side $\mathbf{b}$ the system of linear interval equations $\mathbf{A}x = \mathbf{b}$ is solvable, and hence the system (8), (9) has a solution for each $y \in Y_m$. But, as shown in Theorem 2.2 in [8], in this case we can do essentially better; namely, if we impose an additional complementarity constraint, then the solution turns out to be unique.

THEOREM 6. *Let* $\mathbf{A}$ *be regular. Then for each* $y \in Y_m$ *the system*

$$(A_c - T_y\Delta)x^1 - (A_c + T_y\Delta)x^2 = b_c + T_y\delta, \tag{22}$$

$$x^1 \geq 0, \qquad x^2 \geq 0, \tag{23}$$

$$(x^1)^T x^2 = 0, \tag{24}$$

has a unique solution x_y^1, x_y^2, *and for the solution set* X *of* $\mathbf{A}x = \mathbf{b}$ *defined by* (6) *we have*

$$\operatorname{Conv} X = \operatorname{Conv}\{x_y^1 - x_y^2;\ y \in Y_m\}. \tag{25}$$

Because of (24), for each $y \in Y_m$ the system (22)–(24) can be equivalently written as

$$A_c x - T_y\Delta|x| = b_c + T_y\delta$$

and its unique solution x_y satisfies $x_y = x_y^1 - x_y^2$, so that (25) takes the form

$$\operatorname{Conv} X = \operatorname{Conv}\{x_y;\ y \in Y_m\}. \tag{26}$$

This is the form used in [8]. Theorem 6 has important theoretical consequences. If $[\underline{x}, \overline{x}]$ is the interval hull (optimal enclosure) of the solution set X, then (26) gives

$$\underline{x}_i = \min_{y \in Y_m} (x_y)_i,$$
$$\overline{x}_i = \max_{y \in Y_m} (x_y)_i$$

for $i = 1, \ldots, n$. This result forms a basis for several enclosure algorithms; see [8] and [10].

The number of systems (8), (9) to be checked for solvability is exponential in the number of rows of $\mathbf{A}$ in general. This characterization is unlikely to be substantially improved because of the following complexity result.

THEOREM 7. *Checking solvability of linear interval equations is NP-hard.*

The proof follows easily from the fact that checking regularity of interval matrices, which is an NP-complete problem as proved in [6], can obviously be reduced in polynomial time to the problem of checking solvability of linear interval equations, which is thus NP-hard. NP-hardness of checking nonnegative solvability was established in part 2 of the proof of the main result in [11].

It is instructive to compare the main result of Theorem 3 with its counterpart valid for linear interval inequalities. Analogously to the terminology in section 3, we call a system of linear interval inequalities

$$\mathbf{A}x \leq \mathbf{b}$$

solvable if each system $Ax \leq b$ with $A \in \mathbf{A}$, $b \in \mathbf{b}$ has a solution. Yet the characterization in this case, as shown by Rohn and Kreslová [12], is qualitatively different: although the proof of the "only if" part follows rather similar lines as the respective part of the proof of Theorem 3, it turns out that only *one* system of linear inequalities is to be checked for solvability.

THEOREM 8. *A system of linear interval inequalities* $\mathbf{A}x \leq \mathbf{b}$ *is solvable if and only if the system*

$$\overline{A}x^1 - \underline{A}x^2 \leq \underline{b},$$

$$x^1 \geq 0, \qquad x^2 \geq 0,$$

has a solution.

As a byproduct of the proof we obtain a nontrivial fact which is worth mentioning explicitly [12].

THEOREM 9. *A system of linear interval inequalities* $\mathbf{A}x \leq \mathbf{b}$ *is solvable if and only if all the systems* $Ax \leq b$, $A \in \mathbf{A}$, $b \in \mathbf{b}$, *have a solution in common.*

Based on this comparison, we can conclude that, as regards solvability, linear interval equations and linear interval inequalities behave differently. In the case of exact data, a system of linear equations

$$Ax = b \tag{27}$$

can be equivalently written as

$$\begin{pmatrix} A \\ -A \end{pmatrix} x \leq \begin{pmatrix} b \\ -b \end{pmatrix}, \tag{28}$$

and hence any algorithm for checking solvability of (28) can be employed for checking solvability of (27). This is no more true in the case of inexact data: A system

$$\mathbf{A}x = \mathbf{b} \tag{29}$$

cannot be equivalently written as

$$\begin{pmatrix} \mathbf{A} \\ -\mathbf{A} \end{pmatrix} x \leq \begin{pmatrix} \mathbf{b} \\ -\mathbf{b} \end{pmatrix} \tag{30}$$

because of dependence of data in (28) which is not reflected in (30), where the same coefficient (say, a_{ij}) is allowed to take on different values within its two occurrences. Hence the solution set of (29) is always a part of that of (30), but the converse inclusion need not be true.

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