A sufficient condition for an interval matrix to have full column rank

J. Rohn

Institute of Computer Science, Czech Academy of Sciences, Prague, 18207, Czech Republic
e-mail: rohn@cs.cas.cz

We propose a new sufficient condition for an interval matrix to have full column rank which generalizes the former one based on Beeck's regularity criterion. Combining the new condition with the former one, we get a "double condition" for full rank that has an increased strength. The results of computer experiments are presented that show efficiency of the new full rank test.

Keywords: interval matrix, full column rank, sufficient condition, double condition.

Introduction

By definition, a matrix $B \in \mathbb{R}^{m\times n}$ has *full column rank* if its columns are linearly independent, i. e., if $Bx = 0$ implies $x = 0$. An interval $m \times n$-matrix

$$[A - D, A + D] = \{ B \mid A - D \le B \le A + D \} \tag{1}$$

is said to have full column rank if each $B \in [A - D, A + D]$ has full column rank. In the square case, interval matrices having full column rank are usually called regular, and are said to be singular otherwise.

The problem of checking whether a rectangular interval matrix has full column rank is NP-hard; in fact, it is NP-hard even in the square case [1]. Therefore, it is important to have some verifiable sufficient conditions at hand. Among those surveyed by Shary [2], the following one seems to be most promising: if

$$\rho(|A^+|D) < 1, \tag{2}$$

then $[A - D, A + D]$ has full column rank (where A^+ is the pseudoinverse of A, see below). This assertion was published by the author in [3, p. 34] without proof; a proof has been supplied later by Shary in [2].

In this paper, we first give the original unpublished proof of the assertion and then we formulate another sufficient condition: if

$$\rho\big(\,|(A^TA)^{-1}|D^TD\,\big) < 1, \tag{3}$$

then $[A - D, A + D]$ has full column rank. Because the matrix A^TA is square, in which case inverse and pseudoinverse are identical, we can see that (3) is nothing else than the condition (2) applied to the square interval matrix

$$\big[\,A^TA - D^TD, A^TA + D^TD\,\big]. \tag{4}$$

Thus to carry out the proof of the second condition, it is sufficient to prove that if (4) is regular, then (1) has full column rank, which is proved as an auxiliary result in Theorem 2. The two conditions (2), (3) are then conveniently merged into a single condition

$$\min\{\,\rho(|A^+|D),\ \rho(|(A^TA)^{-1}|D^TD)\,\} < 1. \tag{5}$$

This constitutes the subject matter of Sections 1 and 2. In Section 3, we assess the strength of the joint condition (5), which we call a double condition, on a set of 10 000 randomly generated examples.

We use the following notation. Inequalities and absolute values are understood entrywise. $\rho(A)$ stands for the spectral radius of A and I for the unit matrix. Eigenvalues of a symmetric $n \times n$-matrix A are ordered in nonincreasing way:

$$\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A),$$

and we define the vector of eigenvalues as $\lambda(A) = (\lambda_i(A))_{i=1}^n$. A symmetric matrix A is positive semidefinite if and only if $\lambda_n(A) \geq 0$, and it is positive definite if and only if $\lambda_n(A) > 0$. For the specific case of symmetric matrices, continuity of eigenvalues follows from the Hoffman—Wielandt theorem (see [4, p. 407]):

Theorem 1. *If $A, B \in \mathbb{R}^{n\times n}$ are symmetric, then*

$$\|\lambda(A) - \lambda(B)\|_2 \leq \|A - B\|_F. \tag{6}$$

Here we use the Euclidean vector norm

$$\|x\|_2 = \left(\sum_i x_i^2\right)^{1/2}$$

and the Frobenius matrix norm

$$\|A\|_F = \left(\sum_{i,j} a_{ij}^2\right)^{1/2}.$$

Notice that (6) implies that

$$|\lambda_k(A) - \lambda_k(B)| \leq \|A - B\|_F, \quad k = 1, \ldots, n.$$

For $A \in \mathbb{R}^{m\times n}$, A^+ denotes the unique matrix in $\mathbb{R}^{n\times m}$ satisfying the four conditions:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (A^+A)^T = A^+A, \quad (AA^+)^T = AA^+.$$

It is called the (Moore—Penrose) pseudoinverse of A; see [4] for details. If A has full column rank, then $A^+ = (A^TA)^{-1}A^T$, so that $A^+A = I$ in this case. In particular, if A is square nonsingular, then $A^+ = A^{-1}$.

1. Auxiliary result

The following auxiliary result explains the reasons for taking the interval matrix $[A^TA - D^TD, A^TA + D^TD]$ into consideration.

Theorem 2. *Let $A, D \in \mathbb{R}^{m\times n}$, $D \geq 0$. If the square interval matrix*

$$[A^TA - D^TD, A^TA + D^TD] \tag{7}$$

is regular, then the interval matrix

$$[A - D, A + D] \tag{8}$$

has full column rank.

Proof. Assume to the contrary that (8) does not have full column rank, so that $Cx = 0$, $x \neq 0$, holds for some $C \in [A - D, A + D]$. Then we have

$$|Ax| = |(A - C)x| \leq |A - C||x| \leq D|x|,$$

and consequently

$$x^TA^TAx = (Ax)^T(Ax) \leq |Ax|^T|Ax| \leq (D|x|)^T(D|x|) = |x|^TD^TD|x|. \tag{9}$$

Let L be the diagonal matrix with $L_{ii} = 1$ if $x_i \geq 0$ and $L_{ii} = -1$ otherwise, $i = 1, \ldots, n$. Then $|x| = Lx$ and substituting into (9) we obtain that

$$x^T(A^TA - LD^TDL)x \leq 0.$$

Since $x \neq 0$, this means that the symmetric matrix $A^TA - LD^TDL$ is not positive definite, hence $\lambda_n(A^TA - LD^TDL) \leq 0$.

Now define a real function of one real variable by

$$f(t) = \lambda_n(A^TA - tLD^TDL), \quad t \in [0, 1].$$

It is well defined because $A^TA - tLD^TDL$ is symmetric for each $t \in [0, 1]$, and the above reasoning implies that $f(1) \leq 0$. Next, we have $f(0) = \lambda_n(A^TA) \geq 0$ because A^TA is symmetric positive semidefinite.

The function f is continuous in $[0, 1]$, since for each $t_1, t_2 \in [0, 1]$ we have by the Hoffman—Wielandt theorem

$$|f(t_1) - f(t_2)| \leq \|(t_1 - t_2)LD^TDL\|_F \leq n\|D^TD\|_F|t_1 - t_2|$$

in view of $\|L\|_F = n^{1/2}$. Hence, the intermediate value theorem implies existence of a $t_0 \in [0, 1]$, such that $f(t_0) = 0$. The latter means that the symmetric matrix $S = A^TA - t_0LD^TDL$ satisfies $\lambda_n(S) = 0$ and $|A^TA - S| \leq D^TD$, hence the interval matrix (7) contains a singular symmetric positive semidefinite matrix contrary to the assumption. This contradiction shows that (8) has full column rank. ■

We have proved that regularity of (7) is sufficient for (8) to have full column rank. However, it is not necessary as the following example shows.

Example 1. For
$$A = \begin{pmatrix} 3 & 0 \\ 10 & -8 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$
the interval matrix
$$[A-D, A+D] = \begin{pmatrix} [2,4] & [-1,1] \\ [9,11] & [-9,-7] \end{pmatrix}$$
is regular (i. e., it has full column rank), because for each $A' \in [A-D, A+D]$:
$$\det(A') \in [2,4]\cdot[-9,-7] - [9,11]\cdot[-1,1] = [-36,-14] - [-11,11] = [-47,-3].$$
However, the interval matrix
$$[A^TA - D^TD, A^TA + D^TD] = \begin{pmatrix} [107,111] & [-82,-78] \\ [-82,-78] & [62,66] \end{pmatrix}$$
is singular because it contains a singular matrix
$$S = \begin{pmatrix} 107 & -80.9024 \\ -82 & 62 \end{pmatrix}.$$

A special case of the theorem (for $[A-D, A+D]$ square) was proved in [5].

2. The condition

The following sufficient "double condition" is the main result of this paper.

Theorem 3. *Let $A, D \in \mathbb{R}^{m\times n}$, $D \geq 0$. If A has full column rank and*
$$\min\{\, \rho(|A^+|D),\ \rho(|(A^TA)^{-1}|D^TD) \,\} < 1 \tag{10}$$
holds, then the interval matrix
$$[A-D, A+D] \tag{11}$$
has full column rank.

Proof. Obviously, (10) holds if and only if either
$$\rho(|A^+|D) < 1, \tag{12}$$
or
$$\rho\big(\,|(A^TA)^{-1}|D^TD\,\big) < 1. \tag{13}$$
Thus we must prove that each of the conditions (12), (13) implies that (11) has full column rank.

(a) Let (12) hold. Assume to the contrary that (11) does not have full column rank, so that $Cx = 0$, $x \neq 0$ for some $C \in [A-D, A+D]$. Because A has full column rank, it satisfies $A^+A = I$ and we have
$$|x| = |A^+Ax| \leq |A^+|\,|Ax| = |A^+|\,|(A-C)x| \leq |A^+|\,D\,|x|,$$

which gives

$$(I - |A^+|D)|x| \leq 0. \tag{14}$$

As is well known (Horn and Johnson [4]), the condition (12) implies that $(I - |A^+|D)^{-1} \geq 0$. Premultiplying (14) by this matrix gives $|x| \leq 0$, hence $x = 0$, a contradiction.

(b) Let (13) hold. Because A has full column rank, $A^T A$ is square symmetric positive definite, hence nonsingular. Its inverse is thus equal to its pseudoinverse: $(A^T A)^{-1} = (A^T A)^+$. In this way, we may rewrite (13) as

$$\rho(|(A^T A)^+|D^T D) < 1,$$

and one can immediately see that this is a condition of type (12) for the interval matrix $[A^T A - D^T D, A^T A + D^T D]$.

By what has been proved under (a), we obtain that $[A^T A - D^T D, A^T A + D^T D]$ has full column rank, i. e. it is regular (since it is square), and Theorem 2 asserts that $[A - D, A + D]$ has full column rank. ■

We shall get an explicit version for square matrices when replacing full column rank by nonsingularity and the pseudoinverse by the inverse.

Theorem 4. *Let $A, D \in \mathbb{R}^{n \times n}$, $D \geq 0$. If A is nonsingular and*

$$\min\{\, \rho(|A^{-1}|D),\ \rho(|(A^T A)^{-1}|D^T D) \,\} < 1$$

holds, then the interval matrix

$$[A - D, A + D]$$

is regular.

This is a generalization of Beeck's sufficient condition for regularity of interval matrices [6]. Again, the condition is sufficient, but not necessary.

Example 2. For

$$A = \begin{pmatrix} -1 & -2 \\ -2 & 9 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

the interval matrix $[A - D, A + D]$ is regular, but

$$\min\{\, \rho(|A^{-1}|D),\ \rho(|(A^T A)^{-1}|D^T D) \,\} = \min\{1.0769, 1.4438\} = 1.0769 > 1.$$

3. Condition strength

Denote

$$\begin{aligned} r &= \rho(|A^+|D), \\ s &= \rho(|(A^T A)^{-1}|D^T D). \end{aligned}$$

To assess the strength of the double condition (10) on a set of randomly generated examples, we wrote the following program in MATLAB R2014a:

```
function [rr,ss,tt,fcr,def] = DoubleSuffCondFCR(max,P)
% max maximum size, P number of examples
rr = 0; ss = 0; tt = 0; fcr = 0; def = 0;
for i = 1:P
    rand('state',i);
    m = 2 + floor(rand(1)*(max-1));
    n = 2 + floor(rand(1)*(m-1));
    Ac = 2*rand(m,n) - 1;
    while rank(Ac)<n,
        Ac = 2*rand(m,n) - 1;
    end
    Delta = rand(m,n)/n;
    r = rho(abs(pinv(Ac))*Delta);
    s = rho(abs(inv(Ac'*Ac))*(Delta'*Delta));
    if r<1, rr = rr + 1; end
    if s<1, ss = ss + 1; end
    if r<1 && s < 1, tt = tt + 1; end
    [x] = fullcolrank(Ac,Delta);
    if ~isempty(x)
        def = def + 1;
    else
        fcr = fcr+1;
    end
end

function rh = rho(A)
rh = max(abs(eig(A)));
```

and then we ran it using

```
>> [rr,ss,tt,fcr,def] = DoubleSuffCondFCR(12,10000)
```

The results were as follows.

```
rr =
        3508
ss =
        4212
tt =
        3285
fcr =
        6023
def =
        3977
```

The program generated 10 000 random $m \times n$ interval matrices with $2 \leq n \leq m \leq 12$. Of these 10 000 interval matrices, in 3285 cases there were $r < 1$ and $s < 1$ (hence both constituents of the double condition (10) indicated full column rank), in $3508 - 3285 = 223$

cases there was $r < 1 \leq s$ (only the first condition was met) and in $4212 - 3285 = 927$ cases we had $s < 1 \leq r$ (only the second condition was met). Thus full column rank was detected by the double condition (10) in $3285 + 223 + 927 = 4435$ of the 10 000 cases.

In order to find the exact ratio, we need to find the exact number of full column rank interval matrices among the 10 000 interval matrices generated. This has been done by employing an exponential algorithm `fullcolrank.m` available from [7] which requires solving 2^n linear programming problems for establishing full column rank of an $m \times n$ interval matrix (this is why we imposed the upper bound $n \leq 12$). As it can be seen from the above output, 6023 interval matrices were found to have full column rank, and 3977 to be rank deficient. Thus the double sufficient condition detected full column rank in 4435 of 6023 cases, amounting to 73.6 % successfulness for our sample of 10 000 interval matrices.

This looks perhaps nice, but we should keep in mind that the results may heavily depend on the way in which the random interval matrices are generated. For instance, if we replaced the line

```
Delta = rand(m,n)/n;
```

in the above program by

```
Delta = rand(m,n);
```

then we would have faced an essentially different result

```
rr =
   100
ss =
    82
tt =
    60
fcr =
         1101
def =
         8899
```

where the ratio of successfulness of the double sufficient condition would be 122/1101, i. e., only 11.1 %. Notice also that the ratio of rank deficiency would increase from 39.8 % to 89.0 % as a result of enlarging the radius matrices.

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