A Note on Regularity and Positive Definiteness of Interval Matrices

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Abstract We present a sufficient regularity condition for interval matrices which generalizes two previously known ones. It is formulated in terms of positive definiteness of a certain point matrix, and can also be used for checking positive definiteness of interval matrices. Comparing it with Beeck's strong regularity condition, we show by counterexamples that none of the two conditions is more general than the other one.

Keywords Interval matrix · regularity condition · positive definiteness.

1 Introduction and notation

A square interval matrix

$$\mathbf{A} = [A_c - \Delta, A_c + \Delta] = \{ A \mid A_c - \Delta \le A \le A_c + \Delta \}$$

is called *regular* if each $A \in \mathbf{A}$ is nonsingular, and is said to be *singular* otherwise (i.e., if it contains a singular matrix). The problem of checking regularity of interval matrices is known to be NP-hard [6], which, roughly said, means that existence of a polynomial-time algorithm for its solution is very unlikely because it would imply existence of polynomial-time algorithms for thousands of so-called NP-complete problems [3] for none of which such a polynomial-time algorithm has been found so far despite immense efforts of thousands of computer sciencists over the last 40 years. And indeed, forty

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necessary and sufficient regularity conditions have been found so far [9], all of which exhibit, in some form or another, exponential behavior. This underlines the importance of studying *sufficient* regularity conditions.

In view of what has been said above, one could expect existence of many sufficient regularity conditions. But, surprisingly, the converse is true: only three of them, listed below, are known, at least to these authors.

Theorem 1 *Each of the three conditions implies regularity of* $[A_c - \Delta, A_c + \Delta]$*:*

(i) $\varrho(|A_c^{-1}|\Delta) < 1$,
(ii) $\|\Delta\|_2 < \sigma_{\min}(A_c)$,
(iii) the matrix $A_c^T A_c - \|\Delta^T \Delta\| I$ *is positive definite for some consistent matrix norm* $\|\cdot\|$.

The condition (i) is due to Beeck [1], (ii) is due to Rump [10, Thm. 1.8], and (iii) is due to Rex and Rohn [7, Thm. 5.1]. In (i) ϱ denotes the spectral radius, in (ii) $\sigma_{\min}$ is the minimum singular value, and

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^T A)},$$

where $\lambda_{\max}$, $\sigma_{\max}$ denote the maximum eigenvalue and maximum singular value, respectively. Under a consistent matrix norm in (iii) we understand a matrix norm satisfying $\|AB\| \le \|A\|\|B\|$ for each A, B; I denotes the identity matrix of the respective size. We shall later use the following well-known relationship (to be found e.g. in Horn and Johnson [4]).

Theorem 2 *For each rectangular matrix* A *and each consistent matrix norm* $\|\cdot\|$ *there holds* $\|A\|_2^2 \le \|A^T A\|$.

The following important characterization of singularity of interval matrices is a consequence of the Oettli-Prager theorem [5]; the currently used version can be found e.g. in [2, Thm. 2.9].

Theorem 3 *An interval matrix* $[A_c - \Delta, A_c + \Delta]$ *is singular if and only if the inequality*

$$|A_c x| \le \Delta |x|$$

has a nontrivial solution.

Since $0 \le a \le b$ implies $\|a\|_2 \le \|b\|_2$, we have this corollary.

Corollary 1 *If an interval matrix* $[A_c - \Delta, A_c + \Delta]$ *is singular, then there exists a vector* $x \neq 0$ *satisfying*

$$\|A_c x\|_2 \le \|\Delta |x|\|_2. \tag{1}$$

In fact the condition (iii) of Theorem 1 represents infinitely many conditions depending on the choice of the consistent norm. It is our goal to show that (iii) can be specified in such a way that the resulting condition generalizes not only all the former conditions (iii), but also the condition (ii).

2 New sufficient regularity condition

In this section we present the main result of this paper. The following theorem shows that regularity of **A** can be described in terms of positive definiteness of the matrix (2).

Theorem 4 *Let the matrix*

$$A_c^T A_c - \|\Delta\|_2^2 I \tag{2}$$

be positive definite. Then $[A_c - \Delta, A_c + \Delta]$ *is regular.*

Proof Assume to the contrary that $[A_c - \Delta, A_c + \Delta]$ is singular. Then Corollary 1 implies existence of some $x_0 \neq 0$ such that

$$x_0^T A_c^T A_c x_0 = \|A_c x_0\|_2^2 \leq \|\Delta |x_0|\|_2^2 \leq \|\Delta\|_2^2 (x_0^T x_0),$$

hence we have

$$x_0^T (A_c^T A_c - \|\Delta\|_2^2 I) x_0 \leq 0$$

which means that the matrix (2) is not positive definite, a contradiction. □

3 New condition as a generalization of two older ones

In this section we show that Theorem 4 offers a unified view of two earlier published results. It will be shown that it generalizes not only the regularity condition due to Rump [10], but also all the former regularity conditions due to Rex and Rohn [7].

Theorem 5 *If*

$$\|\Delta\|_2 < \sigma_{\min}(A_c)$$

holds, then the matrix

$$A_c^T A_c - \|\Delta\|_2^2 I$$

is positive definite.

Proof Assume to the contrary that the matrix $A_c^T A_c - \|\Delta\|_2^2 I$ is not positive definite, then there exists an $x_0 \neq 0$ satisfying

$$x_0^T (A_c^T A_c - \|\Delta\|_2^2 I) x_0 \leq 0$$

which can be normalized so that $\|x_0\|_2 = 1$. Consequently

$$\sigma_{\min}^2(A_c) = \lambda_{\min}(A_c^T A_c) = \min_{\|x\|_2=1} x^T A_c^T A_c x \leq x_0^T A_c^T A_c x_0 \leq \|\Delta\|_2^2,$$

hence

$$\sigma_{\min}(A_c) \leq \|\Delta\|_2,$$

which is a contradiction. □

Theorem 6 *If the matrix*

$$A_c^T A_c - \|\Delta^T \Delta\| I \tag{3}$$

is positive definite for some consistent matrix norm $\|\cdot\|$, *then the matrix*

$$A_c^T A_c - \|\Delta\|_2^2 I \tag{4}$$

is positive definite.

Proof Let (3) be positive definite for some consistent matrix norm. Now using Theorem 2, for each $x \neq 0$ we have

$$\begin{aligned} x^T(A_c^T A_c - \|\Delta\|_2^2 I)x &= x^T A_c^T A_c x - \|\Delta\|_2^2 \|x\|_2^2 \\ &\geq x^T A_c^T A_c x - \|\Delta^T \Delta\| \|x\|_2^2 \\ &\geq x^T(A_c^T A_c - \|\Delta^T \Delta\| I)x > 0, \end{aligned}$$

so that the matrix (4) is positive definite and the proof is complete. □

Next we employ the sufficient regularity condition of Theorem 4 for checking positive definiteness of interval matrices.

4 Positive definiteness of interval matrices

Definition 1 A square interval matrix $\mathbf{A}$ is called *symmetric* if $\mathbf{A}^T = \mathbf{A}$, where

$$\mathbf{A}^T = \{A^T \mid A \in \mathbf{A}\}.$$

It can be easily seen that $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is symmetric if and only if both A_c and Δ are symmetric. But, generally, a symmetric interval matrix may contain nonsymmetric point matrices as well.

Definition 2 A symmetric interval matrix is said to be *positive definite* if each symmetric $A \in \mathbf{A}$ is positive definite.

Now we have this characterization.

Theorem 7 *A symmetric interval matrix $\mathbf{A} = [A_c - \Delta, A_c + \Delta]$ is positive definite if and only if*

$$x^T A_c x - |x|^T \Delta |x| > 0 \tag{5}$$

holds for each $x \neq 0$.

Proof First we prove that if (5) holds, then each symmetric $A \in \mathbf{A}$ is positive definite. We show that for each $A \in \mathbf{A}$ and each $x \neq 0$ there holds

$$x^T A x \geq x^T A_c x - |x|^T \Delta |x|.$$

Assume to the contrary that

$$x_0^T A_0 x_0 < x_0^T A_c x_0 - |x_0|^T \Delta |x_0|$$

for some $A_0 \in \mathbf{A}$ and $x_0 \neq 0$. This would imply

$$|x_0|^T \Delta |x_0| < x_0^T (A_c - A_0) x_0 \leq |x_0|^T |A_c - A_0| |x_0| \leq |x_0|^T \Delta |x_0|,$$

a contradiction. Hence the interval matrix $\mathbf{A}$ is positive definite.

Conversely, we are to prove that positive definiteness of all symmetric matrices $A \in \mathbf{A}$ implies that (5) holds for each $x \neq 0$. So let $x \neq 0$ and define a diagonal matrix T as follows: $T_{ii} = 1$ if $x_i \geq 0$, and $T_{ii} = -1$ otherwise $(i = 1, \ldots, n)$, then $Tx = |x|$, and let $A^* = A_c - T\Delta T$. Then A^* is symmetric because A_c, Δ, and T are symmetric, and

$$|A^* - A_c| = |T\Delta T| = \Delta,$$

which means that $A^* \in \mathbf{A}$, so that A^* is positive definite. Now we have

$$0 < x^T A^* x = x^T (A_c - T\Delta T)x = x^T A_c x - x^T T\Delta T x = x^T A_c x - |x|^T \Delta |x|,$$

which was to be proved. □

The proof also yields the following result.

Theorem 8 *If a symmetric interval matrix* $\mathbf{A}$ *is positive definite, then*

$$x^T A x > 0$$

holds for each nonsymmetric $A \in \mathbf{A}$ *and each* $x \neq 0$.

Proof If $\mathbf{A}$ is positive definite, according to Theorem 7, (5) holds, and it was shown in the first part of its proof that this implies $x^T A x > 0$ for each $A \in \mathbf{A}$ and each $x \neq 0$. Let us emphasize that symmetry of A was not assumed in the first part of the proof. □

Hence, nonsymmetric matrices are also "positive definite" except that the term does not apply to them. Now, as soon as we have a tool for checking regularity we can use it for checking positive definiteness of interval matrices. The following link between positive definiteness and regularity of interval matrices was established in [8, Thm. 3].

Theorem 9 *A symmetric interval matrix* $[A_c - \Delta, A_c + \Delta]$ *is positive definite if and only if it is regular and* A_c *is positive definite.*

Using this link, we can turn our sufficient regularity condition into a sufficient positive definiteness condition.

Theorem 10 *Let* $[A_c - \Delta, A_c + \Delta]$ *be symmetric and let both the matrices* A_c *and* $A_c^T A_c - \|\Delta\|_2^2 I$ *be positive definite. Then* $[A_c - \Delta, A_c + \Delta]$ *is positive definite.*

Proof According to Theorem 4 positive definiteness of $A_c^T A_c - \|\Delta\|_2^2 I$ guarantees that $[A_c - \Delta, A_c + \Delta]$ is regular. Also, A_c is positive definite. Now using Theorem 9 gives that $[A_c - \Delta, A_c + \Delta]$ is also positive definite, which was to be proved. □

That means, checking the two mentioned point matrices for positive definiteness suffices to verify positive definiteness of the whole interval matrix. So far, we have studied regularity and positive definiteness of interval matrices. These results will now be utilized to prove the last result of this paper. First we prepare the stage by proving this corollary.

Corollary 2 *An interval matrix of the form* $[A_c^T A_c - \Delta^T \Delta, A_c^T A_c + \Delta^T \Delta]$ *is positive definite if and only if*

$$\|A_c x\|_2 > \|\Delta |x|\|_2 \tag{6}$$

holds for each $x \neq 0$.

Proof Applying Theorem 7 to the interval matrix $[A_c^T A_c - \Delta^T \Delta, A_c^T A_c + \Delta^T \Delta]$ we obtain that it is positive definite if and only if

$$x^T A_c^T A_c x - |x|^T \Delta^T \Delta |x| > 0$$

holds for each $x \neq 0$. Since

$$x^T A_c^T A_c x - |x|^T \Delta^T \Delta |x| = (A_c x)^T (A_c x) - (\Delta |x|)^T (\Delta |x|) = \|A_c x\|_2^2 - \|\Delta |x|\|_2^2,$$

this implies that the above condition is equivalent to

$$\|A_c x\|_2 > \|\Delta |x|\|_2$$

for each $x \neq 0$. So the proof is complete. □

The result is clear: relations (1) and (6) contradict each other. This contradiction leads us to our last result.

Theorem 11 *If* $\mathbf{A} = [A_c^T A_c - \Delta^T \Delta, A_c^T A_c + \Delta^T \Delta]$ *is regular, then* $[A_c - \Delta, A_c + \Delta]$ *is also regular.*

Proof Regularity of $\mathbf{A}$ implies that each $A \in \mathbf{A}$ is nonsingular. So $A_c^T A_c$ is nonsingular. Also it is obvious that $A_c^T A_c$ is positive definite. Thus Theorem 9 gives that $\mathbf{A}$ is positive definite, hence (6) holds by Corollary 2 for each $x \neq 0$. Now assume to the contrary that $[A_c - \Delta, A_c + \Delta]$ is singular. Then (1) holds by Corollary 1 for some $x_0 \neq 0$, a contradiction. This contradiction shows that $[A_c - \Delta, A_c + \Delta]$ is regular as well. □

5 Comparison with the strong regularity condition

In Section 3 we proved that our new sufficient condition of Theorem 4 generalizes the earlier sufficient conditions (ii) and (iii) of Theorem 1. Finally we compare it with Beeck's condition (i) (also called the *strong regularity condition*) and we show by two counterexamples computed in MATLAB that neither of the two conditions is a generalization of the other one. In both examples we use `rand('state',i)` (with $i = 21$ in the first one and $i = 72$ in the second one), so that the data may be reproduced in full precision.

```
n=3; rand('state',21); Ac=2*rand(n,n)-1; Delta=(1/n)*rand(n,n);
A=Ac'*Ac-norm(Delta,2)^2*eye(size(Ac,1)); midrad(Ac,Delta)
rho=max(abs(eig(abs(inv(Ac))*Delta))), eiv=min(eig(A))

intval ans =
[    0.5247,    0.6063] [    0.5343,    0.5599] [   -0.6093,   -0.5652]
[    0.6003,    1.2387] [    0.4443,    0.5948] [    0.0391,    0.2357]
[   -0.7952,   -0.5000] [   -0.1003,    0.1598] [    0.1859,    0.6221]
rho =
    0.9711
eiv =
   -0.0273
```

Here $\varrho(|A_c^{-1}|\Delta) = 0.9711 < 1$ and $\lambda_{\min}(A_c^T A_c - \|\Delta\|_2^2 I) = -0.0273 < 0$, hence the strong regularity condition is satisfied whereas the matrix $A_c^T A_c - \|\Delta\|_2^2 I$ is not positive definite.

```
n=3; rand('state',72); Ac=2*rand(n,n)-1; Delta=(1/n)*rand(n,n);
A=Ac'*Ac-norm(Delta,2)^2*eye(size(Ac,1)); midrad(Ac,Delta)
rho=max(abs(eig(abs(inv(Ac))*Delta))), eiv=min(eig(A))

intval ans =
[   -0.6089,   -0.2581] [   -1.2267,   -0.7475] [   -0.5973,   -0.2492]
[   -0.0397,    0.1292] [   -0.6346,   -0.0022] [    0.3064,    0.8378]
[   -0.9808,   -0.5854] [    0.6140,    1.1957] [    0.5602,    0.6420]
rho =
    1.0254
eiv =
    0.0321
```

Here $\varrho(|A_c^{-1}|\Delta) = 1.0254 > 1$ and $\lambda_{\min}(A_c^T A_c - \|\Delta\|_2^2 I) = 0.0321 > 0$, hence the strong regularity condition is violated whereas the matrix $A_c^T A_c - \|\Delta\|_2^2 I$ is positive definite.

These results finally show that neither of the two conditions can be replaced by the other one, so that we recommend them to be used in conjunction.

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