Optimization Letters manuscript No.
(will be inserted by the editor)

A Theorem of the Alternatives for the Equation $|Ax| - |B||x| = b$

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Received: date / Accepted: date

Abstract A theorem of the alternatives for the equation $|Ax| - |B||x| = b$ ($A, B \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$) is proved and several consequences are drawn. In particular, a class of matrices A, B is identified for which the equation has exactly 2^n solutions for each positive right-hand side b.

Keywords Absolute value equation · triple absolute value equation · alternatives · solution set · interval matrix · regularity.

1 Introduction

We consider here the equation

$$|Ax| - |B||x| = b, \tag{1}$$

where $A, B \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, which we call a *triple absolute value equation.* This equation could also be written in the form

$$|Ax| - C|x| = b,$$

$$C \geq 0,$$

but we prefer the one-line expression (1). As far as known to us, nobody has studied this equation as yet.

In the main result of this paper we show that for each $A, B \in \mathbb{R}^{n\times n}$ exactly one of the following two alternatives holds: (i) for each $b > 0$ the equation (1) has exactly 2^n solutions and the set $\{\, Ax\,;\, |Ax| - |B||x| = b \,\}$ intersects interiors of all orthants

Supported by the Czech Republic Grant Agency under grants 201/09/1957 and 201/08/J020, and by the Institutional Research Plan AV0Z10300504.

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of $\mathbb{R}^n$, (ii) the equation (1) has a nontrivial solution for some $b \leq 0$. In Corollary 1 we show that, even more, if the property mentioned in (i) holds for *some* $b_0 > 0$, then it is shared by *any* $b > 0$, and in Corollary 2 we prove that if A is nonsingular and the condition

$$\varrho(|A^{-1}||B|) < 1 \tag{2}$$

is satisfied (where ϱ stands for the spectral radius), then (i) holds, so that for each $b > 0$ the equation (1) has exactly 2^n solutions. As it will be shown later, these results follow from necessary and/or sufficient conditions for regularity/singularity of interval matrices when applied to the interval matrix $[A - |B|, A + |B|]$. In turn, our results enable us to add two more such necessary and sufficient conditions to the list of forty of them surveyed in [11] (Proposition 1 below).

Nearest in form to the equation (1) is the *absolute value equation*

$$Ax + B|x| = b \tag{3}$$

which has been resently studied by Mangasarian [2], [3], [4], Mangasarian and Meyer [5], Prokopyev [7], and Rohn [10], [12]. There is, however, a big difference between these two equations: while the equation (3) has under the condition (2) exactly one solution for each b (as it follows from Proposition 4.2 in [10] since the condition (2) implies regularity of the interval matrix $[A - |B|, A + |B|]$ as proved in [1]), the equation (1) under the same condition has exactly 2^n solutions for each $b > 0$. This sharp difference between both the equations is to be ascribed to the absence/presence of the absolute value of the term Ax.

The particular circumstances of discovery of the main theorem are briefly mentioned in the personal note in Section 6.

2 Notation

We use the following notation. Matrix inequalities, as $A \leq B$ or $A < B$, are understood componentwise. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (|a_{ij}|)$. The same notation also applies to vectors that are considered one-column matrices. For each $y \in \{-1, 1\}^n$ we denote

$$T_y = \operatorname{diag}(y_1, \ldots, y_n) = \begin{pmatrix} y_1 & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & y_n \end{pmatrix},$$

and $\mathbb{R}^n_y = \{x \; ; \; T_y x \geq 0\}$ is the orthant prescribed by the ± 1-vector y. Notice that $T_y^{-1} = T_y$ for such a y. Given $A, B \in \mathbb{R}^{n \times n}$, the set

$$[A - |B|, A + |B|] = \{\, S \,; \, |S - A| \leq |B| \,\}$$

is an interval matrix; it is called regular if each $S \in [A - |B|, A + |B|]$ is nonsingular, and it is said to be singular otherwise (i.e., if it contains a singular matrix).

3 Theorem of the alternatives

To simplify formulations, we introduce the following definition.

Definition 1 We say that the equation (1) is *exponentially solvable* for a particular right-hand side b if it has exactly 2^n solutions and the set

$$\{ Ax \,;\, |Ax| - |B||x| = b \} \tag{4}$$

intersects interiors of all orthants of $\mathbb{R}^n$.

The following theorem is the main result of this paper.

Theorem 1 *For each $A, B \in \mathbb{R}^{n\times n}$ exactly one of the following two alternatives holds:*

(i) the equation (1) is exponentially solvable for each $b > 0$,
(ii) the equation (1) has a nontrivial solution for some $b \le 0$.

Proof Consider the following two options for the interval matrix $[A - |B|, A + |B|]$:

(i') $[A - |B|, A + |B|]$ is regular,
(ii') $[A - |B|, A + |B|]$ is singular.

We shall prove that the assertions (i), (ii) are equivalent to (i'), (ii'), respectively. Since exactly one of (i'), (ii') always holds, the same will be true for (i), (ii).

(i)⇒(i'). Let (i) hold. Take any $b_0 > 0$, then, by the assumption (i), for each ± 1-vector $y \in \mathbb{R}^n$ there exists a solution x_y of the equation $|Ax| - |B||x| = b_0$ such that $Ax_y \in \mathbb{R}^n_y$. Since x_y satisfies $|Ax_y| = |B||x_y| + b_0 > |B||x_y|$, the condition (v) of Theorem 3.1 in [9] is met and consequently the interval matrix $[A - |B|, A + |B|]$ is regular.

(i')⇒(i). If (i') holds, then for each ± 1-vector y the interval matrix

$$[A - | - T_y|B||, A + | - T_y|B||] = [A - |B|, A + |B|]$$

is regular, hence by Proposition 4.2 in [10] the equation

$$Ax - T_y|B||x| = T_y b \tag{5}$$

has a unique solution x_y. This x_y then satisfies

$$T_y A x_y - |B||x_y| = b, \tag{6}$$

which implies

$$T_y A x_y = |B||x_y| + b \ge b > 0, \tag{7}$$

hence Ax_y belongs to the interior of $\mathbb{R}^n_y$ and $T_y A x_y = |Ax_y|$, which in view of (6) means that x_y is a solution of (1). Conversely, let x solve (1). Put $y_i = 1$ if $(Ax)_i \ge 0$ and $y_i = -1$ otherwise ($i = 1, \ldots, n$), then $T_y Ax = |Ax|$, so that x is a solution of

$$T_y Ax - |B||x| = b$$

and thus also of (5). Because of the above-stated uniqueness of solution of (5), this implies that $x = x_y$. In this way we have proved that the solution set of (1) consists precisely of the points x_y for all possible ± 1-vectors $y \in \mathbb{R}^n$. Thus to prove that (1) has exactly 2^n solutions, it will suffice to show that all the x_y's are mutually different. To this end, take two ± 1-vectors y and y', $y \neq y'$. Then $y_i y'_i = -1$ for some i. From (7) it follows that $y_i(Ax_y)_i > 0$ and $y'_i(Ax_{y'})_i > 0$ and by multiplication $y_i(Ax_y)_i y'_i(Ax_{y'})_i > 0$, hence $(Ax_y)_i(Ax_{y'})_i < 0$, which clearly shows that $x_y \neq x_{y'}$.

(ii)⇔(ii'). Existence of a nontrivial solution of (1) for some $b \leq 0$ is equivalent to existence of a nontrivial solution of the inequality

$$|Ax| \leq |B||x|, \tag{8}$$

which, by Proposition 2.2 in [10], is in turn equivalent to singularity of the interval matrix $[A - |B|, A + |B|]$.

This proves the theorem. □

4 Consequences

We can draw some consequences from Theorem 1 and its proof.

Corollary 1 *If the equation (1) is exponentially solvable for some* $b_0 > 0$*, then it is exponentially solvable for each* $b > 0$*.*

Proof Indeed, in the proof of Theorem 1, implication "(i)⇒(i')", we showed that exponential solvability of the equation (1) for some $b_0 > 0$ implies regularity of $[A - |B|, A + |B|]$ and thus, by "(i')⇒(i)", also exponential solvability for each $b > 0$. □

Corollary 2 *If* A *is nonsingular and*

$$\varrho(|A^{-1}||B|) < 1 \tag{9}$$

holds, then the equation (1) is exponentially solvable for each $b > 0$*.*

Proof By the well-known Beeck's result in [1], the condition (9) implies regularity of the interval matrix $[A - |B|, A + |B|]$ and thus, by the equivalence "(i)⇔(i')" established in the proof of Theorem 1, it also implies exponential solvability of (1) for each $b > 0$. □

Corollary 3 *If* A *is nonsingular and*

$$\max_j (|A^{-1}||B|)_{jj} \geq 1 \tag{10}$$

holds, then the equation (1) is not exponentially solvable for any $b > 0$*.*

Proof It follows from part (iii) of Corollary 5.1 in [8] that the condition (10) implies singularity of the interval matrix $[A - |B|, A + |B|]$, which, by the proof of Theorem 1 and by Corollary 1, precludes exponential solvability of (1) for any $b > 0$. □

For $A, B \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, denote

$$X(A, B, b) = \{\, x\,;\, |Ax| - |B||x| = b \,\},$$

i.e., the solution set of (1) (attention: not to be confused with (4)). Observe that if $x \in X(A, B, b)$, then $-x \in X(A, B, b)$, hence the solutions appear in $X(A, B, b)$ in pairs $(x, -x)$. Thus, unless $b = 0$, the cardinality of $X(A, B, b)$, if finite, is even.

Corollary 4 *If the equation* $|Ax| - |B||x| = b_0$ *is exponentially solvable for some* $b_0 > 0$*, then for each* $b > 0$ *we have*

$$X(A, B, b) = \{ x_y \, ; \, y \in \{-1, 1\}^n \},$$

where for each $y \in \{-1, 1\}^n$*,* x_y *is the unique solution of the absolute value equation*

$$T_y Ax - |B||x| = b. \tag{11}$$

Proof This has been proved in the "(i')⇒(i)" part of the proof of Theorem 1. □

Corollary 5 *Under the assumptions of Corollary 4, we have* $x_{-y} = -x_y$ *for each* $y \in \{-1, 1\}^n$*.*

Proof Since x_y is a solution of (11), it follows that $-x_y$ solves the equation

$$T_{-y} Ax - |B||x| = b,$$

and in view of the uniqueness of solution of this equation we have that $x_{-y} = -x_y$. □

The equation (11) can be solved in a finite number of steps by a very efficient algorithm **absvaleqn** described in [12]. Corollary (5) reduces the number of x_y's to be computed from 2^n to 2^{n-1} (e.g., it suffices to consider only the y's with $y_n = 1$).

Checking regularity of interval matrices is a co-NP-complete problem [6]. Forty necessary and sufficient regularity conditions were surveyed in [11]; the results of this paper enable us to add two more items to the list.

Proposition 1 *For a square interval matrix* $[A - \varDelta, A + \varDelta]$*, the following assertions are equivalent:*

(a) $[A - \varDelta, A + \varDelta]$ *is regular,*
(b) the equation

$$|Ax| - \varDelta|x| = b \tag{12}$$

is exponentially solvable for each $b > 0$*,*
(c) the equation (12) is exponentially solvable for some right-hand side $b_0 > 0$*.*

Proof In the light of Theorem 1 and Corollary 1 we see that (a)⇒(b)⇒(c)⇒(b)⇒(a) holds, which proves the mutual equivalence of all the assertions. □

5 Conclusion

We have investigated the case of $b > 0$. For a general right-hand side b there seems not to be an easy clue to the cardinality of the solution set of (1). This should be a subject of further research.

6 Personal note

I am a little ashamed to admit that I discovered Theorem 1 during the Christmas Eve mass on December 24, 2006 in St Francis Church in Prague.

7 Acknowledgment

The author wishes to thank both referees for their constructive criticism which resulted in essential improvement of the text of the paper.

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