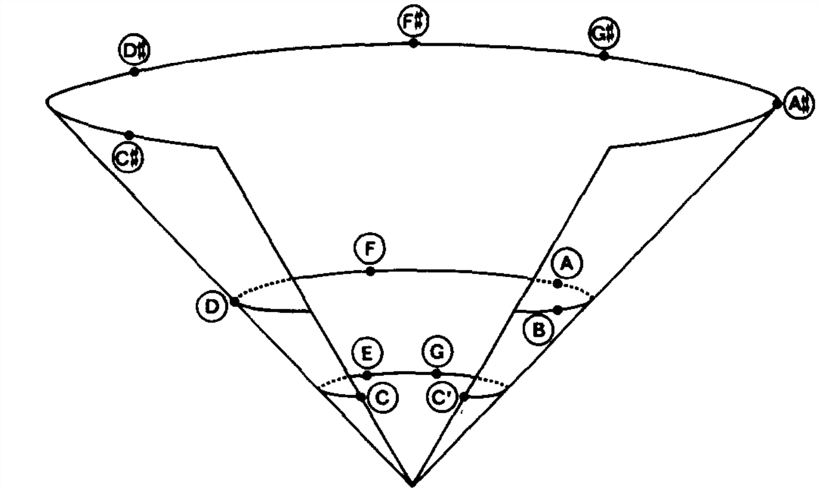
Od hudby přes mozek k diofantským rovnicím

modelování tonality v hudbě pomocí neurálních oscilátorů

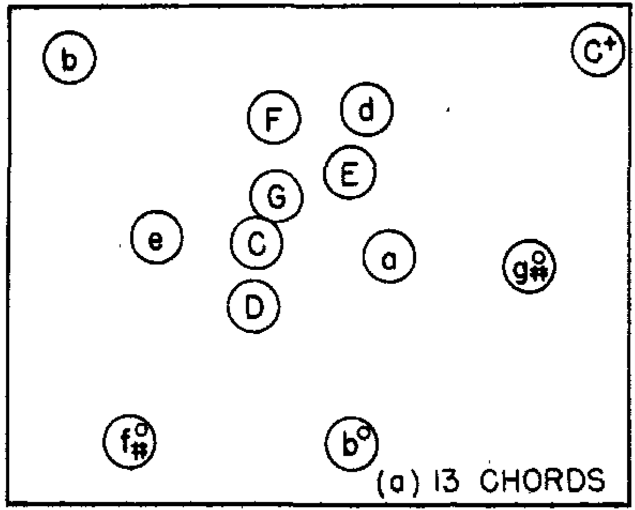
Michal Hadrava

25. května 2017

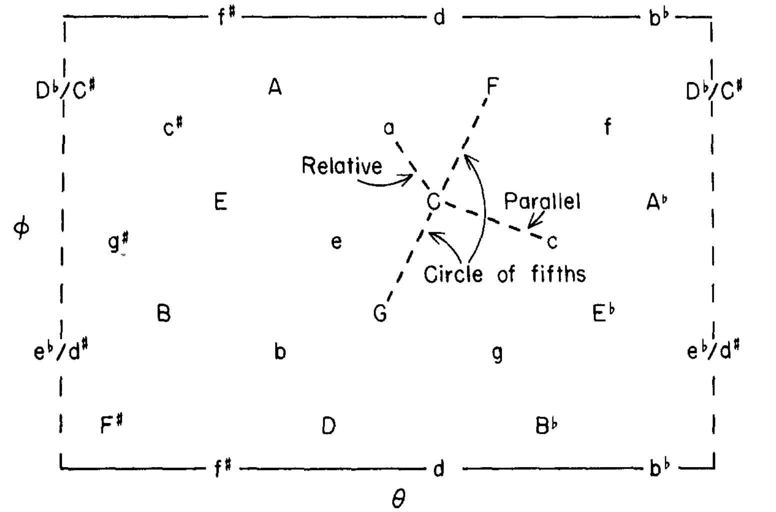
Katedra kybernetiky FEL ČVUT
Ústav informatiky AV ČR
Národní ústav duševního zdraví



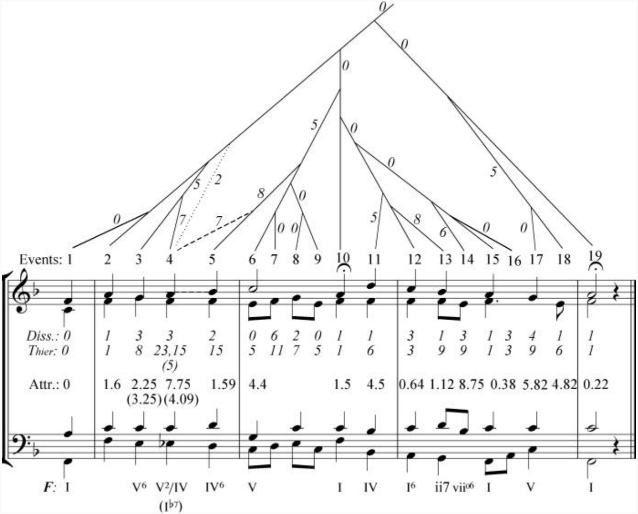
(Krumhansl 1979)



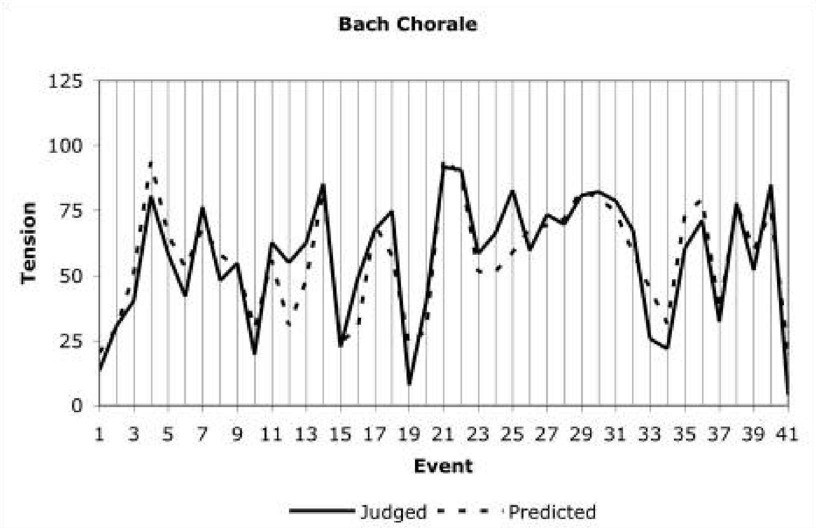
(Krumhansl, Bharucha a Kessler 1982)



(Krumhansl a Kessler 1982)



J. S. Bach (1685 – 1750): "Christus, der is mein Leben", BWV 281 (Lerdahl a Krumhansl 2007).



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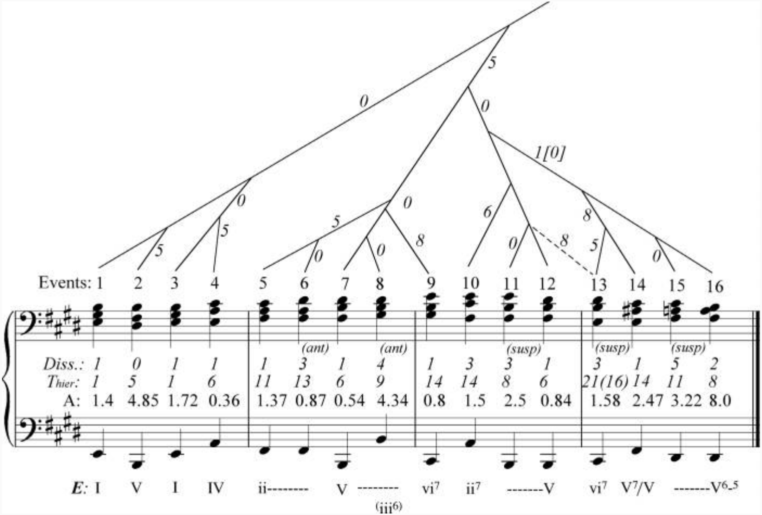
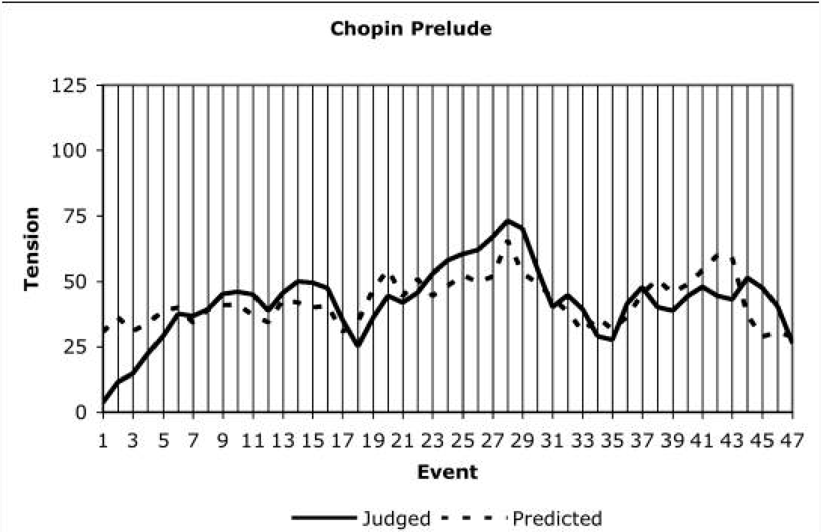


F. Chopin (1810 – 1849): Preludium E dur, Op. 28, No. 9, 1835 – 1839 (Lerdahl a Krumhansl 2007).



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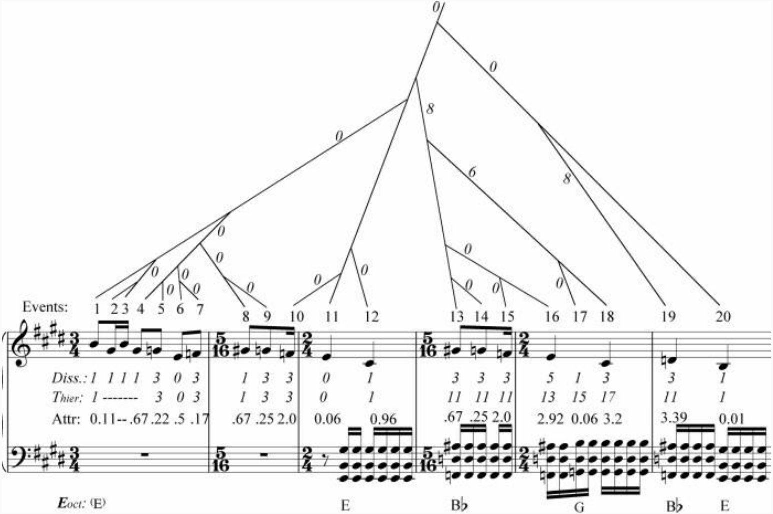
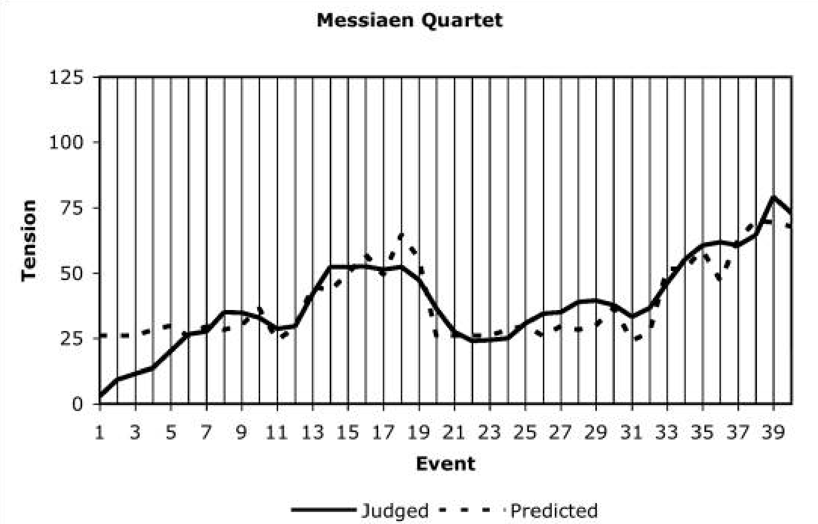


O. Messiaen (1908 – 1992): Quatuor pour la fin du temps, 5. věta, 1940 – 1941 (Lerdahl a Krumhansl 2007).



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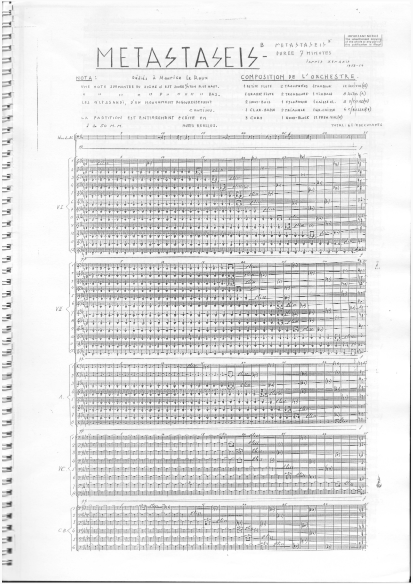
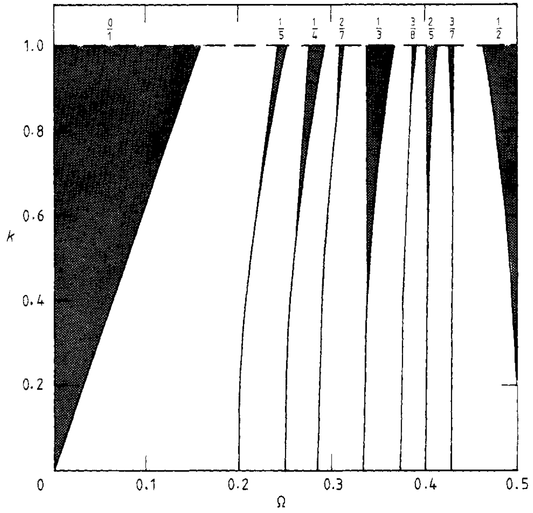
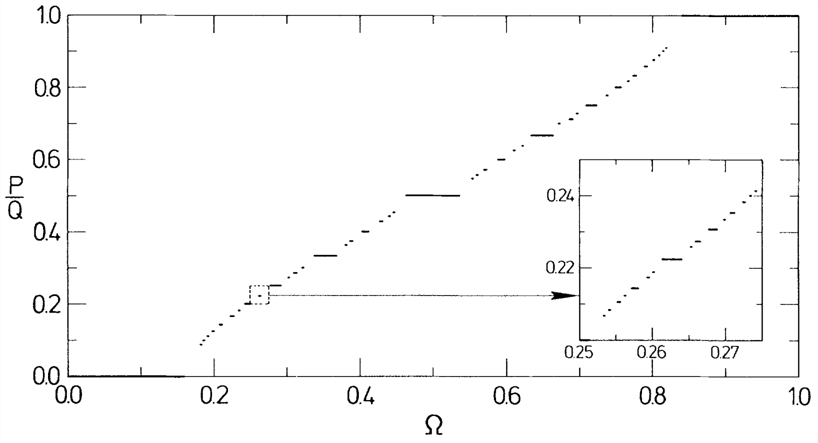


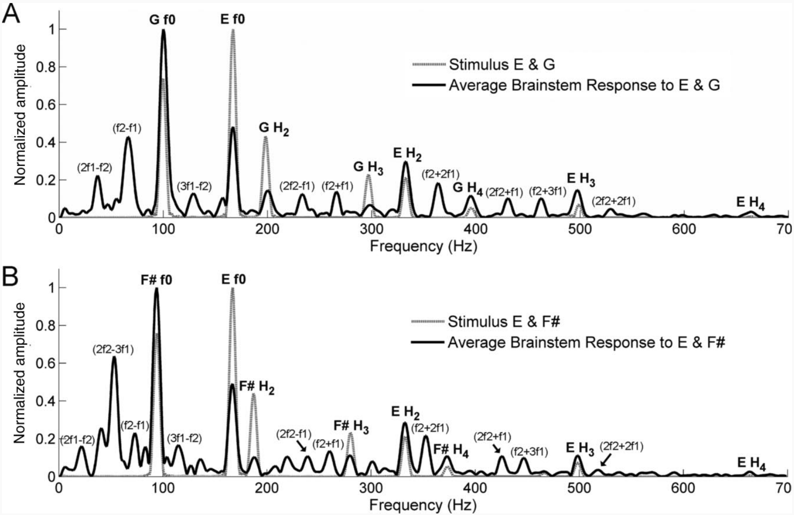
I. Xenakis (1922 – 2001): Metastaseis, 1953 – 1954.



(Ecke, Farmer a Umberger 1989)



(Jensen, Bak a Bohr 1983)



(Lee et al. 2009)

$$\frac{1}{f}\frac{\mathrm{d}u}{\mathrm{d}t} = f(u, v, \boldsymbol{\lambda})$$
$$\frac{1}{f}\frac{\mathrm{d}v}{\mathrm{d}t} = g(u, v, \boldsymbol{\lambda})$$

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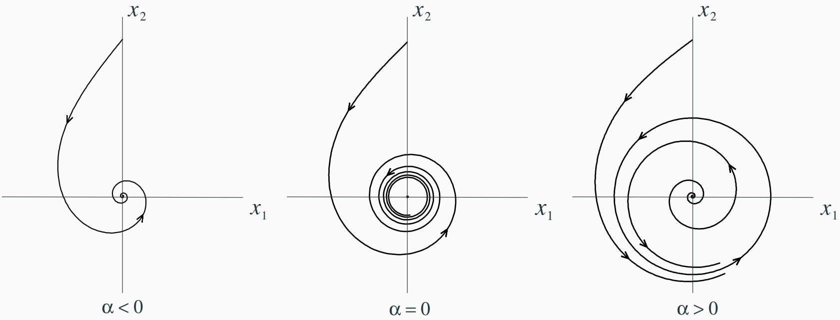
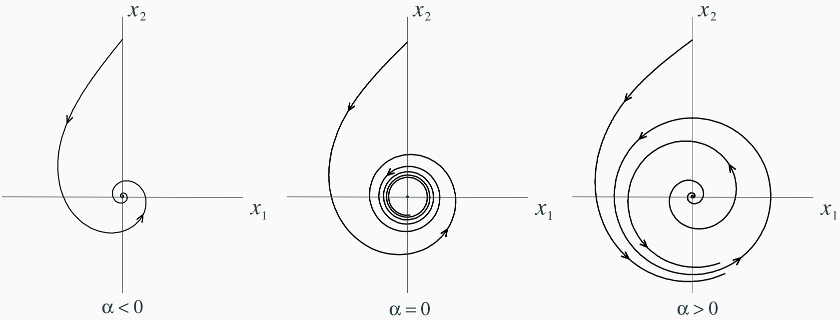


(Kuznetsov 1998)

$$\frac{1}{f}\frac{\mathrm{d}u}{\mathrm{d}t} = f(u, v, \boldsymbol{\lambda}) + \epsilon p(u, v, \boldsymbol{\rho}, \epsilon)$$
$$\frac{1}{f}\frac{\mathrm{d}v}{\mathrm{d}t} = g(u, v, \boldsymbol{\lambda}) + \epsilon q(u, v, \boldsymbol{\rho}, \epsilon)$$



(Kuznetsov 1998)

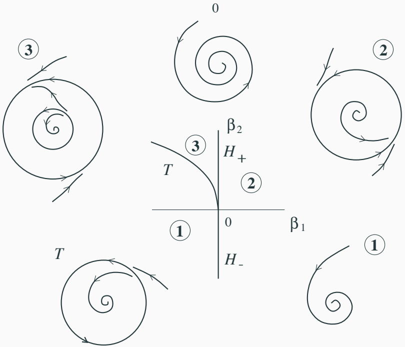
$$
\begin{aligned}
\frac{1}{f}\frac{\mathrm{d}u}{\mathrm{d}t} &= f(u, v, \boldsymbol{\lambda}) + \epsilon p(u, v, \boldsymbol{\rho}, \epsilon) \\
\frac{1}{f}\frac{\mathrm{d}v}{\mathrm{d}t} &= g(u, v, \boldsymbol{\lambda}) + \epsilon q(u, v, \boldsymbol{\rho}, \epsilon)
\end{aligned}
$$

$$\frac{1}{f}\frac{\mathrm{d}u}{\mathrm{d}t} = f(u, v, \boldsymbol{\lambda}) + \epsilon p(u, v, \boldsymbol{\rho}, \epsilon)$$
$$\frac{1}{f}\frac{\mathrm{d}v}{\mathrm{d}t} = g(u, v, \boldsymbol{\lambda}) + \epsilon q(u, v, \boldsymbol{\rho}, \epsilon)$$

$(u, v) \mapsto (z, \bar{z}), \boldsymbol{\lambda} \mapsto (a, b, \boldsymbol{d}), \boldsymbol{\rho} \mapsto (\boldsymbol{x}, \bar{\boldsymbol{x}})$, Taylorův rozvoj

$$\frac{1}{f}\frac{\mathrm{d}z}{\mathrm{d}t} = z(a + b|z|^2 + \sum_{k=0}^{\infty} d_k \epsilon^{k+1}|z|^{2k+4}) + \sum_{\boldsymbol{k}>\mathbf{0}} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(\bar{z} \quad \boldsymbol{x} \quad \bar{\boldsymbol{x}}\right)^{\boldsymbol{k}}$$

$$\frac{1}{f}\frac{\mathrm{d}z}{\mathrm{d}t} = z(a + b|z|^2 + \sum_{k=0}^{\infty} d_k \epsilon^{k+1}|z|^{2k+4}) + \sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(\bar{z} \quad \boldsymbol{x} \quad \bar{\boldsymbol{x}}\right)^{\boldsymbol{k}}$$



(Kuznetsov 1998)

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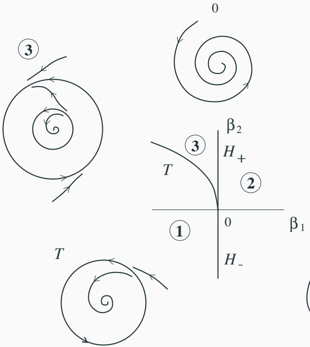


(Kuznetsov 1998)

$$d_k \mapsto d$$

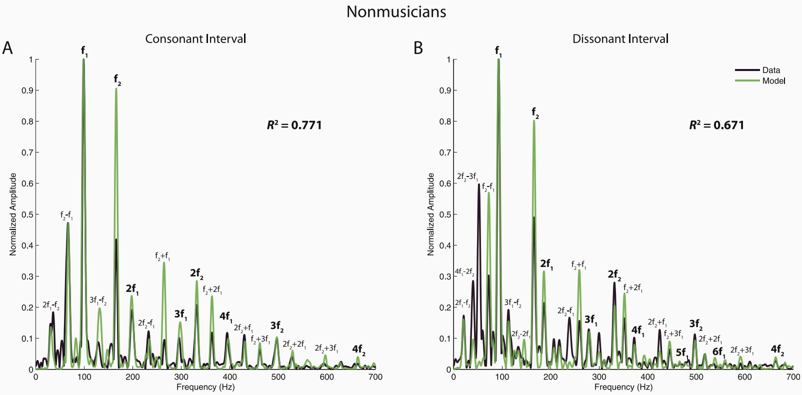
$$\frac{1}{f}\frac{\mathrm{d}z}{\mathrm{d}t} = z(a + b|z|^2 + d\epsilon|z|^4 \sum_{k=0}^{\infty}(\epsilon|z|^2)^k) + \sum_{\mathbf{k}>\mathbf{0}} \sqrt{\epsilon}^{|\mathbf{k}|-1} \left(\bar{z} \quad \boldsymbol{x} \quad \bar{\boldsymbol{x}}\right)^{\mathbf{k}}$$

$$\frac{1}{f}\frac{\mathrm{d}z}{\mathrm{d}t} = z\left(a + b|z|^2 + d\epsilon|z|^4 \sum_{k=0}^{\infty}(\epsilon|z|^2)^k\right) + \sum_{\mathbf{k}>0} \sqrt{\epsilon}^{|\mathbf{k}|-1} \left(\bar{z} \quad x \quad \bar{\mathbf{x}}\right)^{\mathbf{k}}$$

$$\text{geometrická řada } \sum_{k=0}^{\infty}(\epsilon|z|^2)^k = \frac{1}{1-\epsilon|z|^2}, |z| < \sqrt{\frac{1}{\epsilon}}$$

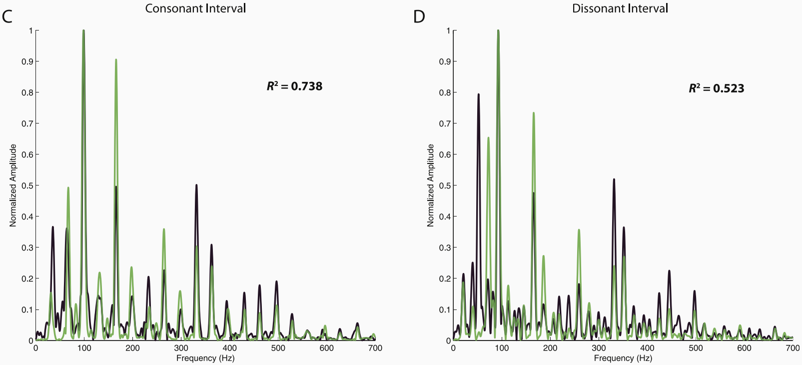
$$\frac{1}{f}\frac{\mathrm{d}z}{\mathrm{d}t} = z\left(a + b|z|^2 + \frac{d\epsilon|z|^4}{1-\epsilon|z|^2}\right) + \sum_{\boldsymbol{k}>\mathbf{0}} \sqrt{\epsilon}^{|\boldsymbol{k}|-1}\left(\bar{z} \quad \boldsymbol{x} \quad \bar{\boldsymbol{x}}\right)^{\boldsymbol{k}}$$

Large, Almonte a Velasco 2010; Lerud et al. 2014

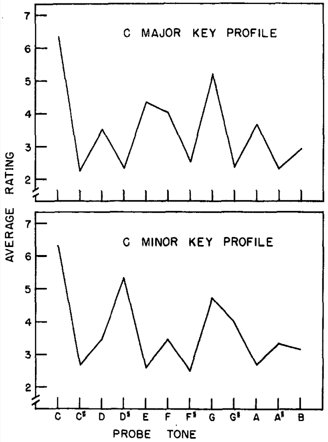


(Lerud et al. 2014)

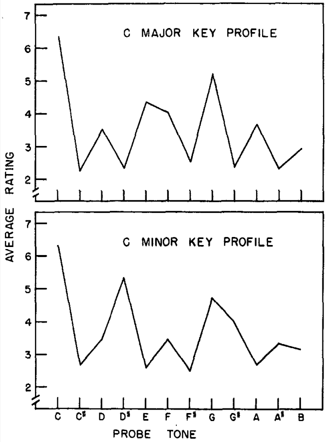
Musicians



(Lerud et al. 2014)



(Krumhansl a Kessler 1982)



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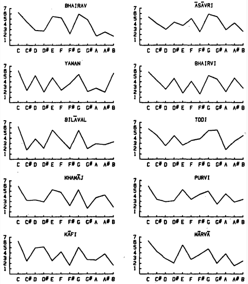
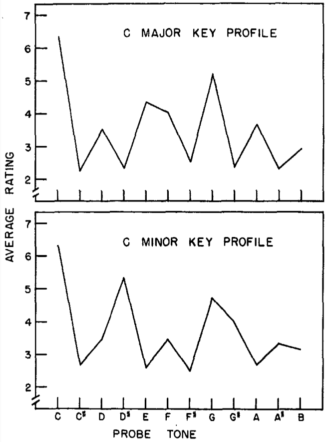


(Castellano, Bharucha a Krumhansl 1984)



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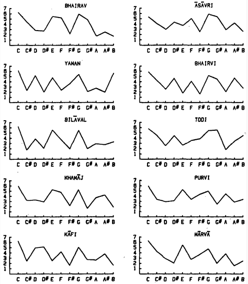

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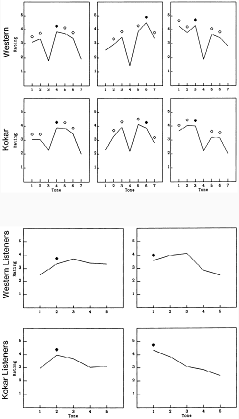


(Kessler, Hansen a Shepard 1984)

$$\frac{1}{f}\frac{\mathrm{d}z}{\mathrm{d}t} = z\left(a + b|z|^2 + \frac{d\epsilon|z|^4}{1-\epsilon|z|^2}\right) + \sum_{\boldsymbol{k}>0}\sqrt{\epsilon}^{|\boldsymbol{k}|-1}\left(\bar{z} \quad \boldsymbol{x} \quad \bar{\boldsymbol{x}}\right)^{\boldsymbol{k}}$$

$$\frac{1}{f}\frac{\mathrm{d}z}{\mathrm{d}t} = z\left(a + b|z|^2 + \frac{d\epsilon|z|^4}{1-\epsilon|z|^2}\right) + \sum_{\boldsymbol{k}>\boldsymbol{0}} \sqrt{\epsilon}^{|\boldsymbol{k}|-1}\left(\bar{z} \quad \boldsymbol{x} \quad \bar{\boldsymbol{x}}\right)^{\boldsymbol{k}}$$

$$z \equiv r\mathrm{e}^{\imath\theta}, \frac{1}{f}\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{e}^{\imath\theta}\left(\frac{1}{f}\frac{\mathrm{d}r}{\mathrm{d}t} + \imath r\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right), \boldsymbol{x} \equiv \boldsymbol{\rho}\mathrm{e}^{\imath\boldsymbol{\theta}}$$

$$
\begin{aligned}
&\mathrm{e}^{\imath\theta}\left(\frac{1}{f}\frac{\mathrm{d}r}{\mathrm{d}t} + \imath r\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = \\
&\quad r\mathrm{e}^{\imath\theta}\left(\alpha + \imath\omega + (\beta_1 + \imath\delta_1)\, r^2 + (\beta_2 + \imath\delta_2)\,\frac{\epsilon r^4}{1-\epsilon r^2}\right) + \\
&\quad \sum_{\boldsymbol{k}>\mathbf{0}} \sqrt{\epsilon}^{|\boldsymbol{k}|-1}\begin{pmatrix} r & \boldsymbol{\rho} & \boldsymbol{\rho} \end{pmatrix}^{\boldsymbol{k}} \mathrm{e}^{\imath\boldsymbol{k}\cdot\begin{pmatrix} -\theta & \boldsymbol{\theta} & -\boldsymbol{\theta} \end{pmatrix}}
\end{aligned}
$$

$$e^{\imath\theta}\left(\frac{1}{f}\frac{\mathrm{d}r}{\mathrm{d}t}+\imath r\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)=$$

$$re^{\imath\theta}\left(\alpha+\imath\omega+(\beta_1+\imath\delta_1)\,r^2+(\beta_2+\imath\delta_2)\,\frac{\epsilon r^4}{1-\epsilon r^2}\right)+$$

$$\sum_{\boldsymbol{k}>\mathbf{0}}\sqrt{\epsilon}^{|\boldsymbol{k}|-1}\begin{pmatrix}r & \boldsymbol{\rho} & \boldsymbol{\rho}\end{pmatrix}^{\boldsymbol{k}} e^{\imath\boldsymbol{k}\cdot\begin{pmatrix}-\theta & \boldsymbol{\theta} & -\boldsymbol{\theta}\end{pmatrix}}$$

$$\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

$$\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1-\epsilon r^2} +$$

$$\frac{1}{r}\sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \Im\left(\mathrm{e}^{\imath\left(\boldsymbol{k}\cdot\left(-\theta \quad \boldsymbol{\theta} \quad -\boldsymbol{\theta}\right)-\theta\right)}\right)$$

$$\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1 - \epsilon r^2} + \frac{1}{r}\sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \begin{pmatrix} r & \boldsymbol{\rho} & \boldsymbol{\rho} \end{pmatrix}^{\boldsymbol{k}} \Im\left(\mathrm{e}^{\imath\left(\boldsymbol{k}\cdot\begin{pmatrix} -\theta & \boldsymbol{\theta} & -\boldsymbol{\theta} \end{pmatrix} - \theta\right)} \right)$$

$$= \omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1 - \epsilon r^2} + \frac{1}{r}\sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \begin{pmatrix} r & \boldsymbol{\rho} & \boldsymbol{\rho} \end{pmatrix}^{\boldsymbol{k}} \sin\left(\boldsymbol{k}\cdot\begin{pmatrix} -\theta & \boldsymbol{\theta} & -\boldsymbol{\theta} \end{pmatrix} - \theta\right)$$

$$
\begin{aligned}
\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t} =& \omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1-\epsilon r^2} + \\
& \frac{1}{r}\sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \begin{pmatrix} r & \boldsymbol{\rho} & \boldsymbol{\rho} \end{pmatrix}^{\boldsymbol{k}} \Im \left(e^{\imath \left(\boldsymbol{k} \cdot \begin{pmatrix} -\theta & \boldsymbol{\theta} & -\boldsymbol{\theta} \end{pmatrix} - \theta \right)} \right) \\
=& \omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1-\epsilon r^2} + \\
& \frac{1}{r}\sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \begin{pmatrix} r & \boldsymbol{\rho} & \boldsymbol{\rho} \end{pmatrix}^{\boldsymbol{k}} \sin \left(\boldsymbol{k} \cdot \begin{pmatrix} -\theta & \boldsymbol{\theta} & -\boldsymbol{\theta} \end{pmatrix} - \theta \right)
\end{aligned}
$$

$$
f \equiv 1, \delta_1 \equiv \delta_2 \equiv 0, \begin{pmatrix} r & \boldsymbol{\rho} & \boldsymbol{\rho} \end{pmatrix}^{\boldsymbol{k}} \mapsto c, \theta_1 \leftrightarrow \theta_2
$$

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \omega_1 + \sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} c \sin\left(\boldsymbol{k} \cdot \begin{pmatrix} -\theta_1 & \theta_2 & -\theta_2 \end{pmatrix} - \theta_1\right)$$

$$\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \omega_2 + \sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} c \sin\left(\boldsymbol{k} \cdot \begin{pmatrix} -\theta_2 & \theta_1 & -\theta_1 \end{pmatrix} - \theta_2\right)$$

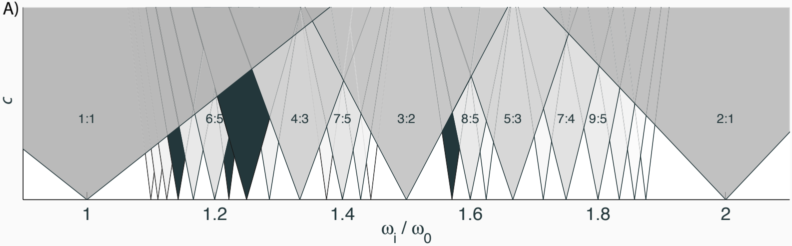
$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \omega_1 + \sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} c \sin\left(\boldsymbol{k} \cdot \begin{pmatrix} -\theta_1 & \theta_2 & -\theta_2 \end{pmatrix} - \theta_1\right)$$

$$\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \omega_2 + \sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} c \sin\left(\boldsymbol{k} \cdot \begin{pmatrix} -\theta_2 & \theta_1 & -\theta_1 \end{pmatrix} - \theta_2\right)$$

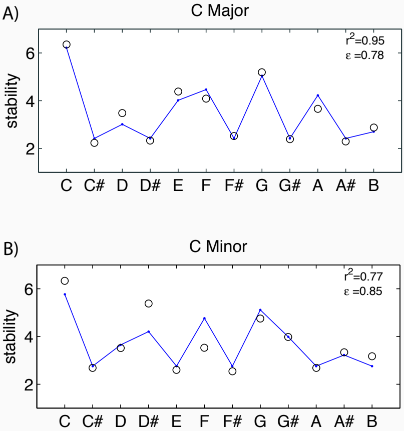
$$\boldsymbol{k} \mapsto \begin{pmatrix} (m-1) & k & 0 \end{pmatrix}, \begin{pmatrix} (k-1) & m & 0 \end{pmatrix}$$

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \omega_1 + \sqrt{\epsilon}^{k+m-2} c \sin(k\theta_2 - m\theta_1)$$
$$\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \omega_2 + \sqrt{\epsilon}^{m+k-2} c \sin(m\theta_1 - k\theta_2)$$

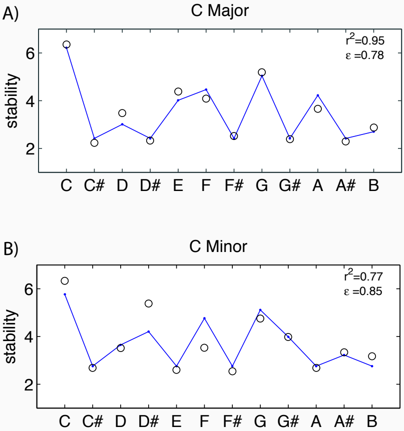
$$\frac{d\theta_1}{dt} = \omega_1 + \sqrt{\epsilon}^{k+m-2} c \sin(k\theta_2 - m\theta_1)$$
$$\frac{d\theta_2}{dt} = \omega_2 + \sqrt{\epsilon}^{m+k-2} c \sin(m\theta_1 - k\theta_2)$$



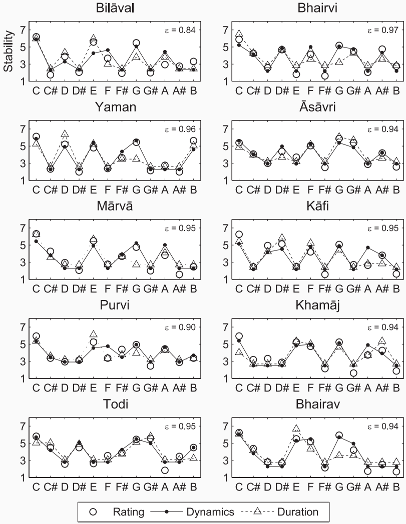
(Large 2010)


A)
C Major
r^2=0.95
ε =0.78
stability
6
4
2
C C# D D# E F F# G G# A A# B
B)
C Minor
r^2=0.77
ε =0.85
stability
6
4
2
C C# D D# E F F# G G# A A# B

(Large 2010)



(Large 2010)



(Large, Kim et al. 2016)

$$\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1-\epsilon r^2} + \dots$$

$$\frac{1}{r}\sum_{\boldsymbol{k}>\boldsymbol{0}} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \sin\left(\boldsymbol{k}\cdot\left(-\theta \quad \boldsymbol{\theta} \quad -\boldsymbol{\theta}\right) - \theta\right)$$

$$\begin{aligned}\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t} =& \omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1-\epsilon r^2} + \dots \\ & \frac{1}{r}\sum_{\boldsymbol{k}>\mathbf{0}} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \sin\left(\boldsymbol{k}\cdot\left(-\theta \quad \boldsymbol{\theta} \quad -\boldsymbol{\theta}\right) - \theta\right) \\ \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} =& \boldsymbol{f}\omega\end{aligned}$$

$$
\begin{aligned}
\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t} =& \omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1-\epsilon r^2} + \dots \\
& \frac{1}{r}\sum_{\boldsymbol{k}>\mathbf{0}} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \sin\left(\boldsymbol{k}\cdot\left(-\theta \quad \boldsymbol{\theta} \quad -\boldsymbol{\theta}\right) - \theta\right) \\
\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} =& \boldsymbol{f}\omega
\end{aligned}
$$

$$
f \equiv \tilde{f}(1+\delta_0), \frac{\boldsymbol{f}}{\tilde{f}} \equiv \tilde{\boldsymbol{f}},
$$

$$\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1-\epsilon r^2} + \dots$$

$$\frac{1}{r}\sum_{\boldsymbol{k}>\boldsymbol{0}} \sqrt{\epsilon}^{|\boldsymbol{k}|-1}\left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \sin\left(\boldsymbol{k}\cdot\left(-\theta \quad \boldsymbol{\theta} \quad -\boldsymbol{\theta}\right) - \theta\right)$$

$$\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} = \boldsymbol{f}\omega$$

$$f \equiv \tilde{f}(1+\delta_0), \frac{\boldsymbol{f}}{\tilde{f}} \equiv \tilde{\boldsymbol{f}},$$

$$\tau \equiv \tilde{f}t, \Delta \equiv \delta_0\omega + (1+\delta_0)\left(\delta_1 r^2 + \delta_2\frac{\epsilon r^4}{1-\epsilon r^2}\right),$$

$$
\begin{aligned}
\frac{1}{f}\frac{\mathrm{d}\theta}{\mathrm{d}t} =&\omega + \delta_1 r^2 + \delta_2 \frac{\epsilon r^4}{1-\epsilon r^2} + \dots \\
&\frac{1}{r}\sum_{\boldsymbol{k}>\boldsymbol{0}} \sqrt{\epsilon}^{|\boldsymbol{k}|-1}\left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \sin\left(\boldsymbol{k}\cdot\left(-\theta \quad \boldsymbol{\theta} \quad -\boldsymbol{\theta}\right) - \theta\right) \\
\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t} =& \boldsymbol{f}\omega
\end{aligned}
$$

$$
\begin{gathered}
f \equiv \tilde{f}(1+\delta_0), \frac{\boldsymbol{f}}{\tilde{f}} \equiv \tilde{\boldsymbol{f}}, \\
\tau \equiv \tilde{f}t, \Delta \equiv \delta_0\omega + (1+\delta_0)\left(\delta_1 r^2 + \delta_2\frac{\epsilon r^4}{1-\epsilon r^2}\right), \\
\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \frac{\mathrm{d}\theta}{\mathrm{d}\tilde{f}t} = \frac{1}{\tilde{f}}\frac{\mathrm{d}\theta}{\mathrm{d}t}, \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}\tau} = \frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}\tilde{f}t} = \frac{1}{\tilde{f}}\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}
\end{gathered}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\theta}{\mathrm{d}\tau} =& \omega + \Delta + \dots \\
& \frac{1+\delta_0}{r} \sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho} \right)^{\boldsymbol{k}} \sin\left(\boldsymbol{k} \cdot \left(-\theta \quad \boldsymbol{\theta} \quad -\boldsymbol{\theta} \right) - \theta \right) \\
\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}\tau} =& \tilde{\boldsymbol{f}} \omega
\end{aligned}
$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \omega + \Delta + \dots$$

$$\frac{1+\delta_0}{r} \sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho} \right)^{\boldsymbol{k}} \sin \left(\boldsymbol{k} \cdot \left(-\theta \quad \boldsymbol{\theta} \quad -\boldsymbol{\theta} \right) - \theta \right)$$

$$\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}\tau} = \tilde{\boldsymbol{f}}\omega$$

$$\left(\phi \quad \boldsymbol{\phi} \right) \equiv \left(\theta \quad \boldsymbol{\theta} \right) - \left(\omega \quad \tilde{\boldsymbol{f}}\omega \right) \tau$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \Delta + \frac{1+\delta_0}{r}\sum_{\boldsymbol{k}>0}\sqrt{\epsilon}^{|\boldsymbol{k}|-1}\left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}}\cdots$$

$$\sin\left(\boldsymbol{k}\cdot\left(-\left(\omega\tau+\phi\right) \quad \left(\tilde{\boldsymbol{f}}\omega\tau+\boldsymbol{\phi}\right) \quad -\left(\tilde{\boldsymbol{f}}\omega\tau+\boldsymbol{\phi}\right)\right)-\left(\omega\tau+\phi\right)\right)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \Delta + \frac{1+\delta_0}{r} \sum_{\boldsymbol{k}>0} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \dots$$

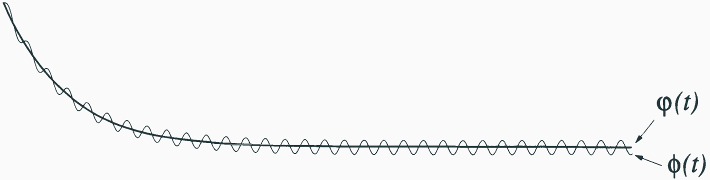
$$\sin\left(\boldsymbol{k} \cdot \left(-(\omega\tau + \phi) \quad \left(\tilde{\boldsymbol{f}}\omega\tau + \boldsymbol{\phi}\right) \quad - \left(\tilde{\boldsymbol{f}}\omega\tau + \boldsymbol{\phi}\right)\right) - (\omega\tau + \phi)\right)$$

$$\varphi \equiv \phi - h\left(\varphi, \boldsymbol{\phi}, \tau\right)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \Delta + \frac{1+\delta_0}{r}\sum_{\boldsymbol{k}>0}\sqrt{\epsilon}^{|\boldsymbol{k}|-1}\left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \dots$$

$$\sin\left(\boldsymbol{k}\cdot\left(-(\omega\tau+\phi) \quad \left(\tilde{\boldsymbol{f}}\omega\tau+\boldsymbol{\phi}\right) \quad -\left(\tilde{\boldsymbol{f}}\omega\tau+\boldsymbol{\phi}\right)\right)-(\omega\tau+\phi)\right)$$

$$\varphi \equiv \phi - h\left(\varphi, \boldsymbol{\phi}, \tau\right)$$



(Hoppensteadt a Izhikevich 1997)

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \approx \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\, \mathrm{d}\tau$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \approx \lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\,\mathrm{d}\tau$$

(Hoppensteadt a Izhikevich 1997, Theorem 9.6)

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \approx \Delta + \frac{1+\delta_0}{r}\sum \sqrt{\epsilon}^{|\boldsymbol{k}|-1}\left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \sin\left(\boldsymbol{k}\cdot\left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi}\right) - \varphi\right)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \approx \Delta + \frac{1+\delta_0}{r} \sum \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}} \sin\left(\boldsymbol{k} \cdot \left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi}\right) - \varphi\right)$$

$$\left(-\tilde{f} \quad \boldsymbol{f} \quad -\boldsymbol{f}\right) \cdot \boldsymbol{k} = \tilde{f}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \approx \Delta + \frac{1+\delta_0}{r} \sum \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \begin{pmatrix} r & \boldsymbol{\rho} & \boldsymbol{\rho} \end{pmatrix}^{\boldsymbol{k}} \sin\left(\boldsymbol{k} \cdot \begin{pmatrix} -\varphi & \boldsymbol{\varphi} & -\boldsymbol{\varphi} \end{pmatrix} - \varphi\right)$$

$$\begin{pmatrix} -\tilde{f} & \boldsymbol{f} & -\boldsymbol{f} \end{pmatrix} \cdot \boldsymbol{k} = \tilde{f}$$

$$S = \{\boldsymbol{m} + \boldsymbol{s} \mid \boldsymbol{m} \in M, \boldsymbol{s} \in S_0\}$$

$$\begin{pmatrix}1 & 4\end{pmatrix} \cdot \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix} - \begin{pmatrix}2 & 3\end{pmatrix} \cdot \begin{pmatrix}\eta_1 & \eta_2\end{pmatrix} \qquad = 4$$

$$\begin{pmatrix}1 & 4\end{pmatrix} \cdot \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix} - \begin{pmatrix}2 & 3\end{pmatrix} \cdot \begin{pmatrix}\eta_1 & \eta_2\end{pmatrix} = 4$$

$$\begin{pmatrix}a_1 & a_2\end{pmatrix} \cdot \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix} - \begin{pmatrix}b_1 & b_2\end{pmatrix} \cdot \begin{pmatrix}\eta_1 & \eta_2\end{pmatrix} = c$$

$$\begin{pmatrix}1 & 4\end{pmatrix} \cdot \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix} - \begin{pmatrix}2 & 3\end{pmatrix} \cdot \begin{pmatrix}\eta_1 & \eta_2\end{pmatrix} = 4$$

$$\begin{pmatrix}a_1 & a_2\end{pmatrix} \cdot \begin{pmatrix}\xi_1 & \xi_2\end{pmatrix} - \begin{pmatrix}b_1 & b_2\end{pmatrix} \cdot \begin{pmatrix}\eta_1 & \eta_2\end{pmatrix} = c$$

	-4	-3	-2	-1	0	1	2	3	4
4						b_2	b_1		
3					b_2	b_1			
2				b_2	b_1				
1			b_2	b_1					
0						a_1			a_2
-1					a_1			a_2	
-2				a_1			a_2		
-3			a_1			a_2			
-4		a_1			a_2				

(Clausen a Fortenbacher 1989)

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \approx \Delta + \frac{1+\delta_0}{r}\sum_{\boldsymbol{k}\in M}\sqrt{\epsilon}^{|\boldsymbol{k}|-1}\left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho}\right)^{\boldsymbol{k}}\sin\left(\boldsymbol{k}\cdot\left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi}\right)-\varphi\right)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} \approx \Delta + \frac{1+\delta_0}{r} \sum_{\boldsymbol{k}\in M} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \left(r \quad \boldsymbol{\rho} \quad \boldsymbol{\rho} \right)^{\boldsymbol{k}} \sin\left(\boldsymbol{k} \cdot \left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi} \right) - \varphi \right)$$

$$\delta_1 \equiv \delta_2 \equiv 0, r \equiv \rho_i \equiv \sqrt{\gamma}$$

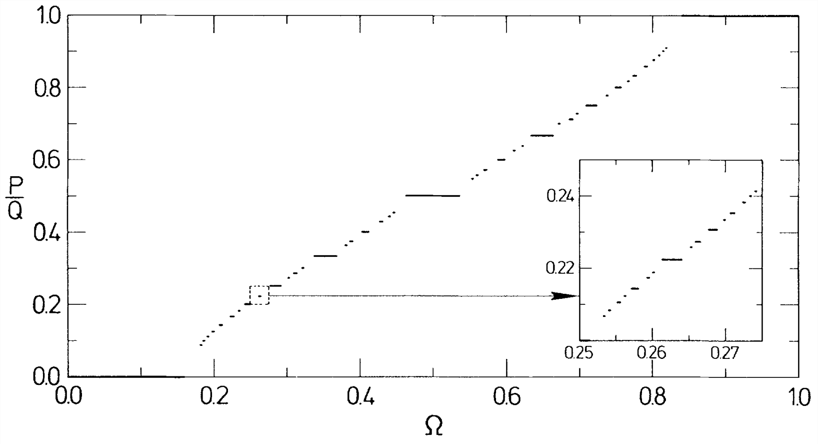
$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \delta_0 \omega + (1 + \delta_0) \sum_{\boldsymbol{k} \in M} \sqrt{\epsilon}^{|\boldsymbol{k}|-1} \sqrt{\gamma}^{|\boldsymbol{k}|-1} \sin\left(\boldsymbol{k} \cdot \left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi} \right) - \varphi \right)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \delta_0\omega + (1+\delta_0)\sum_{\boldsymbol{k}\in M}\sqrt{\epsilon}^{|\boldsymbol{k}|-1}\sqrt{\gamma}^{|\boldsymbol{k}|-1}\sin\left(\boldsymbol{k}\cdot\left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi}\right) - \varphi\right)$$

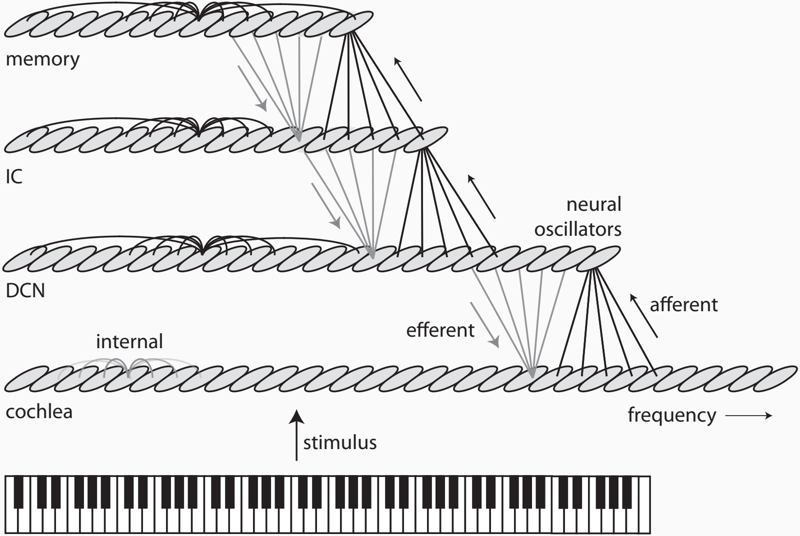
$$\epsilon\gamma \mapsto \varepsilon$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \delta_0\omega + (1+\delta_0)\sum_{\boldsymbol{k}\in M}\sqrt{\varepsilon}^{|\boldsymbol{k}|-1}\sin\left(\boldsymbol{k}\cdot\left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi}\right)-\varphi\right)$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \delta_0\omega + (1+\delta_0)\sum_{\boldsymbol{k}\in M}\sqrt{\varepsilon}^{|\boldsymbol{k}|-1}\sin\left(\boldsymbol{k}\cdot\left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi}\right) - \varphi\right)$$



(Jensen, Bak a Bohr 1983)


memory
IC
neural
oscillators
DCN
afferent
efferent
internal
cochlea
frequency
stimulus

(Large 2011)

(A, A)-*otevřený diskrétní dynamický systém* (Spivak 2016):

- $A \equiv \{(f, \varphi) \mid f \in \mathbb{R}^n_{>0}, \varphi \in \mathbb{T}^n\}$,

(A, A)-*otevřený diskrétní dynamický systém* (Spivak 2016):

- $A \equiv \{(\boldsymbol{f}, \boldsymbol{\varphi}) \mid \boldsymbol{f} \in \mathbb{R}^n_{>0}, \boldsymbol{\varphi} \in \mathbb{T}^n\}$,
- $S \equiv \{(\tilde{f}, \varphi) \mid \tilde{f} \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \varphi \colon \mathbb{R}_{>0} \to \mathbb{T}\}$,

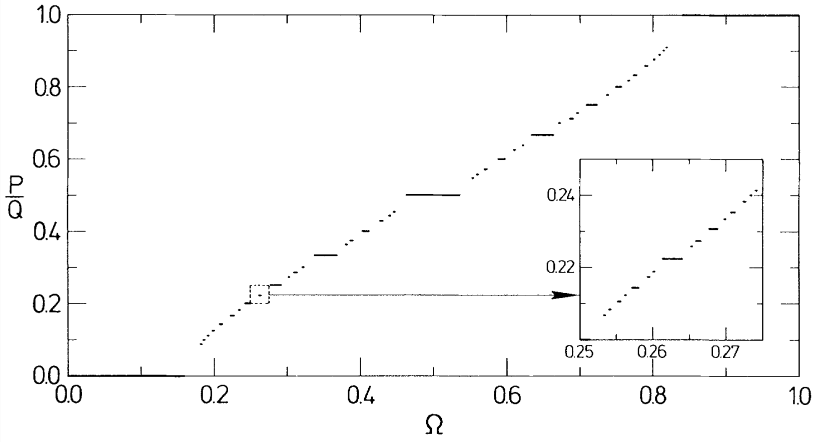
(A, A)-*otevřený diskrétní dynamický systém* (Spivak 2016):

- $A \equiv \{(\boldsymbol{f}, \boldsymbol{\varphi}) \mid \boldsymbol{f} \in \mathbb{R}^n_{>0}, \boldsymbol{\varphi} \in \mathbb{T}^n\}$,
- $S \equiv \{(\tilde{f}, \varphi) \mid \tilde{f} \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \varphi \colon \mathbb{R}_{>0} \to \mathbb{T}\}$,
- $f^{\mathrm{upd}} \colon A^2 \times S \to S$,

(A, A)-*otevřený diskrétní dynamický systém* (Spivak 2016):

- $A \equiv \{(\boldsymbol{f}, \boldsymbol{\varphi}) \mid \boldsymbol{f} \in \mathbb{R}^n_{>0}, \boldsymbol{\varphi} \in \mathbb{T}^n\}$,
- $S \equiv \{(\tilde{f}, \varphi) \mid \tilde{f} \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}, \varphi \colon \mathbb{R}_{>0} \to \mathbb{T}\}$,
- $f^{\mathrm{upd}} \colon A^2 \times S \to S$,
- $f^{\mathrm{rdt}} \colon S \to A$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \delta_0\omega + (1+\delta_0)\sum_{\boldsymbol{k}\in M}\sqrt{\varepsilon}^{|\boldsymbol{k}|-1}\sin\left(\boldsymbol{k}\cdot\left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi}\right) - \varphi\right)$$



(Jensen, Bak a Bohr 1983)

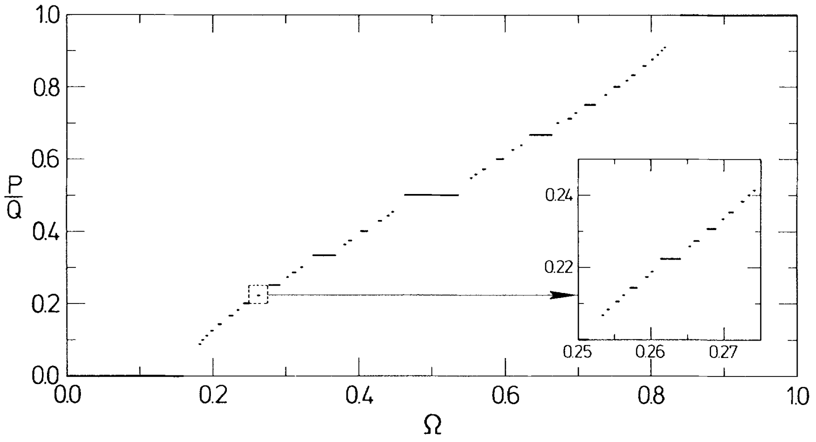
$$J = (1+\delta_0)\sum_{\boldsymbol{k}\in M}\sqrt{\varepsilon}^{|\boldsymbol{k}|-1}(-k_1-1)\cos\left(\boldsymbol{k}\cdot\left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi}\right)-\varphi\right)$$

$$J = (1+\delta_0)\sum_{\boldsymbol{k}\in M} \sqrt{\varepsilon}^{|\boldsymbol{k}|-1}(-k_1-1)\cos\left(\boldsymbol{k}\cdot\begin{pmatrix}-\varphi & \boldsymbol{\varphi} & -\boldsymbol{\varphi}\end{pmatrix} - \varphi\right)$$

	-4	-3	-2	-1	0	1	2	3	4
4						b_2	b_1		
3					b_2	b_1			
2				b_2	b_1				
1			b_2	b_1					
0						a_1			a_2
-1					a_1			a_2	
-2				a_1			a_2		
-3			a_1			a_2			
-4		a_1			a_2				

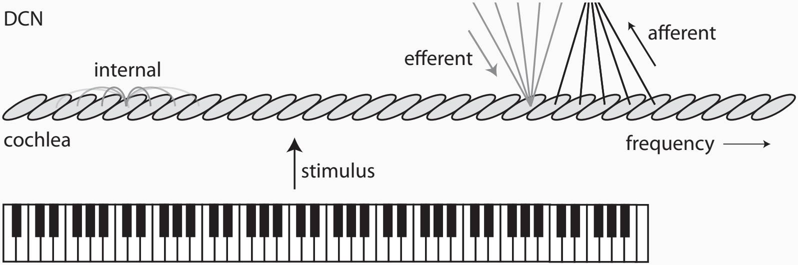
(Clausen a Fortenbacher 1989)

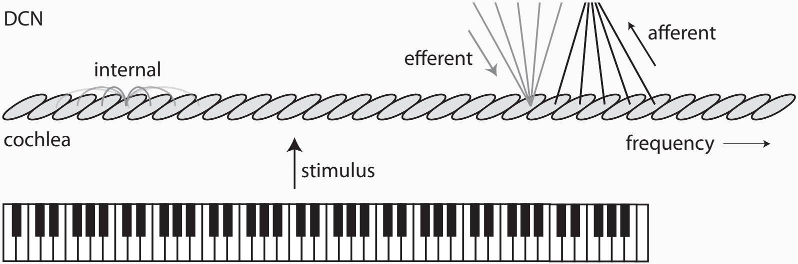
$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \delta_0\omega + (1+\delta_0)\sum_{\boldsymbol{k}\in M}\sqrt{\varepsilon}^{|\boldsymbol{k}|-1}\sin\left(\boldsymbol{k}\cdot\left(-\varphi \quad \boldsymbol{\varphi} \quad -\boldsymbol{\varphi}\right) - \varphi\right)$$



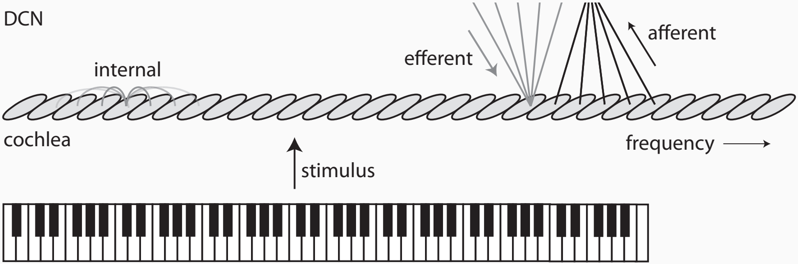
(Jensen, Bak a Bohr 1983)

matice ekvilibrií $M_{ij} \equiv \#\{s \in S \mid f^{\mathrm{rdt}}(s) = j, f^{\mathrm{upd}}(i, s) = s\}$
(Spivak 2016)


DCN
afferent
efferent
internal
cochlea
frequency
stimulus

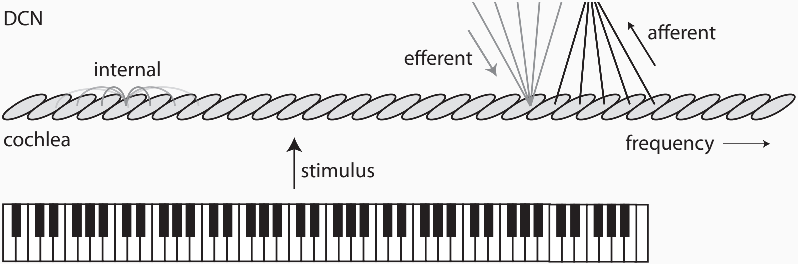


$$\widehat{\varphi^{\mathrm{in}}}\colon ((a_{\mathrm{stim}}, a_{\mathrm{eff}}), a_{\mathrm{aff}}) \mapsto (a_{\mathrm{stim}}, a_{\mathrm{eff}}, a_{\mathrm{aff}})$$



$$\widehat{\varphi^{\mathrm{in}}}\colon ((a_{\mathrm{stim}}, a_{\mathrm{eff}}), a_{\mathrm{aff}}) \mapsto (a_{\mathrm{stim}}, a_{\mathrm{eff}}, a_{\mathrm{aff}})$$

$$\widehat{\varphi^{\mathrm{out}}}\colon a_{\mathrm{aff}} \to a_{\mathrm{aff}}$$



$$\widehat{\varphi^{\text{in}}}\colon ((a_{\text{stim}}, a_{\text{eff}}), a_{\text{aff}}) \mapsto (a_{\text{stim}}, a_{\text{eff}}, a_{\text{aff}})$$

$$\widehat{\varphi^{\text{out}}}\colon a_{\text{aff}} \to a_{\text{aff}}$$

$$N_{ij} = \sum_{k \in \widehat{\varphi^{\text{out}}}^{-1}(j)} M_{\widehat{\varphi^{\text{in}}(i,k)}\,k} = M_{\widehat{\varphi^{\text{in}}(i,j)}\,j}$$

(Spivak 2016)

Mary A. Castellano, Jamshed J. Bharucha a Carol L. Krumhansl. “Tonal hierarchies in the music of north India”. In: *Journal of Experimental Psychology. General* 113.3 (1984), s. 394.

M. Clausen a A. Fortenbacher. “Efficient solution of linear diophantine equations”. In: *Journal of Symbolic Computation* 8.1–2 (čvc—srp. 1989), s. 201–216. DOI: 10.1016/S0747-7171(89)80025-2.

Robert E. Ecke, J. Doyne Farmer a David K. Umberger. “Scaling of the Arnold tongues”. In: *Nonlinearity* 2.2 (1989), s. 175.

Frank C. Hoppensteadt a Eugene M. Izhikevich. *Weakly connected neural networks*. Sv. 126. Applied Mathematical Sciences. Springer-Verlag New York, 1997. DOI: 10.1007/978-1-4612-1828-9.

M. Høgh Jensen, Per Bak a Tomas Bohr. “Complete devil’s staircase, fractal dimension, and universality of mode-locking structure in the circle map”. In: *Physical Review Letters* 50.21 (1983), s. 1637.

Edward J. Kessler, Christa Hansen a Roger N. Shepard. “Tonal schemata in the perception of music in Bali and in the West”. In: *Music Perception. An Interdisciplinary Journal* 2.2 (1984), s. 131–165.

Carol L. Krumhansl. “The psychological representation of musical pitch in a tonal context”. In: *Cognitive Psychology* 11.3 (1979), s. 346–374.

Carol L. Krumhansl, Jamshed J. Bharucha a Edward J. Kessler. “Perceived harmonic structure of chords in three related musical keys”. In: *Journal of Experimental Psychology. Human Perception and Performance* 8.1 (1982), s. 24.

Carol L. Krumhansl a Edward J. Kessler. “Tracing the dynamic changes in perceived tonal organization in a spatial representation of musical keys”. In: *Psychological Review* 89.4 (1982), s. 334.

Yuri A. Kuznetsov. *Elements of applied bifurcation theory*. Springer, 1998.

- Edward W. Large. “A dynamical systems approach to musical tonality”. In: *Nonlinear Dynamics in Human Behavior*. Ed. Raoul Huys a Viktor K. Jirsa. Springer, 2010, s. 193–211.
- Edward W. Large. “Musical tonality, neural resonance and Hebbian learning”. In: *International Conference on Mathematics and Computation in Music*. Springer. 2011, s. 115–125.
- Edward W. Large, Felix V. Almonte a Marc J. Velasco. “A canonical model for gradient frequency neural networks”. In: *Physica D - Nonlinear Phenomena* 239.12 (červ. 2010), s. 905–911. DOI: `10.1016/j.physd.2009.11.015`.

Edward W. Large, Ji Chul Kim et al. “A neurodynamic account of musical tonality”. In: *Music Perception. An Interdisciplinary Journal* 33.3 (2016), s. 319–331.

Kyung Myun Lee et al. “Selective subcortical enhancement of musical intervals in musicians”. In: *Journal of Neuroscience* 29.18 (2009), s. 5832–5840.

Fred Lerdahl a Carol L. Krumhansl. “Modeling tonal tension”. In: *Music Perception. An Interdisciplinary Journal* 24.4 (2007), s. 329–366.

Karl D. Lerud et al. “Mode-locking neurodynamics predict human auditory brainstem responses to musical intervals”. In: *Hearing Research* 308.SI (ún. 2014), s. 41–49. DOI: 10.1016/j.heares.2013.09.010.

David I. Spivak. “The steady states of coupled dynamical systems compose according to matrix arithmetic”. Ver. 2. In: *arXiv preprint arXiv:1512.00802* (24. ún. 2016). arXiv: `1512.00802v2 [math.DS]`. URL: `https://arxiv.org/abs/1512.00802v2` (cit. 05. 04. 2017).