Unification types in modal logics

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Unification in a propositional logic $\mathbf{L}$

Logical unification

- problems of **making given formulas equivalent in** $\mathbf{L}$ by **replacing their variables by formulas**

References

- **Baader, F., Snyder, W. :** *Unification theory.* In : *Handbook of Automated Reasoning*, Elsevier (2001) 439–526.
- **Siekmann, J. :** *Unification theory.* Journal of Symbolic Computation **7** (1989) 207–274.

Introduction

Let us consider a propositional logic **L** like

- Classical Propositional Logic **CPL**, Intuitionistic Propositional Logic **IPL**, ...
- Propositional modal logics **S**4, **S**5, ...
- Description logics $\mathcal{FL}_0$, $\mathcal{EL}$, ...

Unification problems in **L**

Given a pair (φ, ψ) of formulas

- is there a substitution σ such that $\sigma(\varphi) \leftrightarrow \sigma(\psi)$ is in **L** ?

Given finitely many pairs $(\varphi_1, \psi_1), \ldots, (\varphi_n, \psi_n)$ of formulas

- is there a substitution σ such that $\sigma(\varphi_1) \leftrightarrow \sigma(\psi_1)$, ..., $\sigma(\varphi_n) \leftrightarrow \sigma(\psi_n)$ are in **L** ?

Given a formula φ

- is there a substitution σ such that $\sigma(\varphi)$ is in **L** ?

Introduction

Language of **L** : formulas are constructed by means of

- Variables x, y, ...
- Parameters p, q, ...
- Connectives $\bot$, $\top$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\Box$, $\Diamond$, ...

Elementary unification in **L**

- Given a parameter-free formula $\psi(x_1, \ldots, x_n)$
- Determine whether there exists formulas φ_1, ..., φ_n such that $\psi(\varphi_1, \ldots, \varphi_n)$ is in **L**

Unification with parameters in **L**

- Given a formula $\psi(p_1, \ldots, p_m, x_1, \ldots, x_n)$
- Determine whether there exists formulas φ_1, ..., φ_n such that $\psi(p_1, \ldots, p_m, \varphi_1, \ldots, \varphi_n)$ is in **L**

Introduction

In Classical Propositional Logic **CPL**, Intuitionistic Propositional Logic **IPL**, ...

- Elementary unifiability **is equivalent to** consistency
- Why ? Use Uniform Substitution

In propositional modal logic **S**4, **S**5, ...

- Elementary unifiability **is not equivalent to** consistency
- Why ? Consider the formula $\Diamond x \wedge \Diamond \neg x$ and use Uniform Substitution

Introduction

Admissibility in **L**

- The rule of inference $\frac{\varphi_1 \; \cdots \; \varphi_m}{\psi}$ is admissible in **L** if for all substitutions σ, if $\sigma(\varphi_1), \ldots, \sigma(\varphi_m)$ are in **L** then $\sigma(\psi)$ is in **L**

Unifiability by means of admissibility

If L is consistent then the following are equivalent :

- Formula φ is unifiable in **L**
- Rule of inference $\frac{\varphi}{\bot}$ is non-admissible in **L**

Admissibility by means of unifiability

If L is finitary or unitary then the following are equivalent :

- Rule of inference $\frac{\varphi_1 \; \cdots \; \varphi_m}{\psi}$ is admissible in **L**
- Formulas $\sigma(\psi)$ is in **L** for each maximal **L**-unifiers σ of $\varphi_1 \wedge \ldots \wedge \varphi_m$

Introduction

Suppose **L** is **axiomatically presented (axioms + rules of inference)**

Derivability of formulas from hypothesis

- A **derivation** in **L** of formula ψ from hypothesis $\varphi_1, \ldots, \varphi_n$ is a sequence $\psi_1, \ldots, \psi_k$ of formulas such that $\psi = \psi_k$ and for all $i = 1 \ldots k$, at least one of the following conditions holds
  - ψ_i is an instance of an axiom of **L**
  - ψ_i can be obtained from $\psi_1, \ldots, \psi_{i-1}$ by means of at least one of the rules of inference of **L**
  - ψ_i is equal to one of the hypothesis $\varphi_1, \ldots, \varphi_n$
- We write $\varphi_1, \ldots, \varphi_n \vdash_{\mathbf{L}} \psi$

Derivability of inference rules

- The rule of inference $\frac{\varphi_1 \;\; \ldots \;\; \varphi_m}{\psi}$ is **derivable** in **L** if $\varphi_1, \ldots, \varphi_n \vdash_{\mathbf{L}} \psi$

Introduction

Suppose **L** is **axiomatically presented (axioms + rules of inference)**

Proposition

- Every **L**-derivable rule of inference is admissible

Why ? Use Uniform Substitution

Structural completeness

- **L** is said to be **structurally complete** if every **L**-admissible rule of inference is derivable

Introduction

About Classical Propositional Logic **CPL**

CPL is **structurally complete** : every **CPL**-admissible rule of inference is derivable

- Thus, admissibility in **CPL** is decidable
- In fact, in **CPL**, the **admissibility problem** is equivalent to the **derivability problem**

About Intuitionistic Propositional Logic **IPL**

IPL is **not structurally complete** : some **IPL**-admissible rules of inference are not derivable

- $\frac{\neg x \to y \vee z}{(\neg x \to y) \vee (\neg x \to z)}$ — **Harrop rule (1960)**
- $\frac{(\neg\neg x \to x) \to (x \vee \neg x)}{\neg\neg x \vee \neg x}$ — **Lemmon-Scott rule**
- $\frac{(x \to y) \to (x \vee \neg y)}{\neg\neg x \vee \neg y}$ — **generalized Lemmon-Scott rule**
- $\frac{(x \to y) \to x \vee z}{((x \to y) \to x) \vee ((x \to y) \to z)}$ — **Mints rule (1972)**

Introduction

About Intuitionistic Propositional Logic **IPL**

The following rule is admissible in **IPL** but not derivable

- $\frac{(\neg\neg x \to x) \to (x \vee \neg x)}{\neg\neg x \vee \neg x}$ — **Lemmon-Scott rule**

About propositional modal logic **S**4

The following rule is admissible in **S**4 but not derivable

- $\frac{\Box(\Box(\Box\Diamond\Box x \to \Box x) \to (\Box x \vee \Box\neg\Box x))}{\Box\Diamond\Box x \vee \Box\neg\Box x}$ — **modal Lemmon-Scott rule**

Introduction

About intermediate logics

If L is an intermediate logic — IPL $\subseteq$ L $\subseteq$ CPL — then the following are equivalent for each rule of inference $\mathcal{R}$

- $\mathcal{R}$ is admissible in **L**
- The modal translation of $\mathcal{R}$ is admissible in the greatest modal companion of **L**

Reference

- **Rybakov, V. :** *Admissible rules for pretable modal logics.* Algebra and Logic **20** (1981) 291–307.

Introduction

Other references

Rybakov (1981)

- The unification problem and the admissibility problem in extensions of propositional modal logic **S**4.3 are decidable

Rybakov (1984, 1997)

- The unification problem and the admissibility problem in propositional modal logics **K**4, **S**4, . . . are decidable

Chagrov (1992)

- There exists a decidable propositional modal logic with an undecidable admissibility problem

Wolter and Zakharyaschev (2008)

- The unification problem and the admissibility problem for any propositional modal logic between $\mathbf{K}_U$ and $\mathbf{K}4_U$ are undecidable

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Propositional language

Formulas are constructed by means of

- Set **VAR** of propositional variables x, y, $\ldots$
- Set **PAR** of propositional parameters p, q, $\ldots$
- Connectives $\bot$, $\top$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\Box$, $\Diamond$, $\ldots$

For all finite subsets X of **VAR**

- Let $\mathcal{L}_X$ be the set of all formulas φ such that $\mathtt{var}(\varphi) \subseteq X$

Substitutions are functions of the form

- $\sigma : \mathcal{L}_X \longrightarrow \mathcal{L}_Y$

where X, Y are finite subsets of **VAR**

Composition of substitutions

Given $\sigma : \mathcal{L}_X \longrightarrow \mathcal{L}_Y$ and $\tau : \mathcal{L}_Y \longrightarrow \mathcal{L}_Z$, let

- $\tau\sigma : \mathcal{L}_X \longrightarrow \mathcal{L}_Z$ with $(\tau\sigma)(\varphi) = \tau(\sigma(\varphi))$

From now on

Let **L** be a propositional logic

Equivalence between substitutions

Let $\sigma : \mathcal{L}_X \longrightarrow \mathcal{L}_Y$ and $\tau : \mathcal{L}_X \longrightarrow \mathcal{L}_Y$

- $\sigma \simeq_L \tau$ if for all variables x in X, $\sigma(x) \leftrightarrow \tau(x) \in \mathbf{L}$
- "σ and τ are **L**-equivalent"

Preorder between substitutions

Let $\sigma : \mathcal{L}_X \longrightarrow \mathcal{L}_Y$ and $\tau : \mathcal{L}_X \longrightarrow \mathcal{L}_Z$

- $\sigma \preceq_L \tau$ if there exists a substitution $\mu : \mathcal{L}_Y \longrightarrow \mathcal{L}_Z$ such that $\mu\sigma \simeq_L \tau$
- "σ is less specific, more general than τ in **L**"

Unifiers and bases

Unifiers

- A substitution $\sigma : \mathcal{L}_X \longrightarrow \mathcal{L}_Y$ is a unifier of a formula φ in **L** if $X = \mathtt{var}(\varphi)$ and $\sigma(\varphi) \in \mathbf{L}$

Complete sets of unifiers

- A set Σ of unifiers of a formula φ is complete if for all unifiers τ of φ, there exists a unifier σ of φ in Σ such that $\sigma \preceq_L \tau$

Bases

- A complete set Σ of unifiers of a formula is a basis if for all σ, τ in Σ, if $\sigma \preceq_L \tau$ then $\sigma = \tau$

Unifiers and bases

Important property

- All bases of unifiers of a formula have the same cardinality

Important questions

- Given a formula, has it a unifier ?
- If so, has it a basis ?
- If so, how large is this basis ? Is this basis effectively computable ?

Unification problems

Elementary unification in L

input : a parameter-free formula φ

output : determine whether there exists a unifier of φ in **L**

Unification with parameters in L

input : a formula φ

output : determine whether there exists a unifier of φ in **L**

Types of formulas

Nullary formulas

- A **L**-unifiable formula φ is nullary (or of type 0) if it has no basis

Infinitary formulas

- A **L**-unifiable formula φ is infinitary (or of type ∞) if it has an infinite basis

Finitary formulas

- A **L**-unifiable formula φ is finitary (or of type ω) if it has a finite basis of cardinality ≥ 2

Unitary formulas

- A **L**-unifiable formula φ is unitary (or of type 1) if it has a finite basis of cardinality $= 1$

Types of propositional logics

The unification types being ordered by $1 < \omega < \infty < 0$, the unification type of **L** is the greatest type among the types of its unifiable formulas, i.e.

- **L** is nullary (or of type 0) if there exists a nullary **L**-unifiable formula
- **L** is infinitary (or of type ∞) if every **L**-unifiable formula is either infinitary, or finitary, or unitary and there exists an infinitary **L**-unifiable formula
- **L** is finitary (or of type ω) if every **L**-unifiable formula is either finitary, or unitary and there exists a finitary **L**-unifiable formula
- **L** is unitary (or of type 1) if every **L**-unifiable formula is unitary

Propositional modal language

Formulas (φ, ψ, ...) are constructed by means of

- Set **VAR** of propositional variables x, y, ...
- Set **PAR** of propositional parameters p, q, ...
- Connectives $\bot$, $\top$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\Box$, $\Diamond$, ...

Formal definition of the set $\mathcal{L}$ of all formulas

- $\varphi \;::=\; x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$

Abbreviations

- $\top$, $\wedge$, $\rightarrow$, $\leftrightarrow$ are defined as usual
- $\Diamond$ is defined by $\Diamond\varphi \;::=\; \neg\Box\neg\varphi$
- $\Box^{\leq k}\varphi \;::=\; \varphi \wedge \Box\varphi \wedge \ldots \wedge \Box^{k}\varphi$

Propositional modal logic

Set **L** of formulas closed under uniform substitution and such that

1. **L** contains all tautologies
2. **L** contains the distribution axiom : $\Box(x \to y) \to (\Box x \to \Box y)$
3. **L** is closed under modus ponens : $\frac{x \quad x \to y}{y}$
4. **L** is closed under generalization : $\frac{x}{\Box x}$

Examples

- Least modal logic
  - **K**
- Least modal logic containing modal logic **L** and formula φ
  - **L** $+ \varphi$
- Greatest modal logic
  - $\mathcal{L}$ — the only inconsistent modal logic
- Greatest consistent modal logics
  - **Ver** $=$ **K** $+ \Box\bot$
  - **Triv** $=$ **K** $+ \Box x \leftrightarrow x$

Other examples

$\mathbf{KD}$	$=$	$\mathbf{K} + \Box x \to \Diamond x$
$\mathbf{KT}$	$=$	$\mathbf{K} + \Box x \to x$
$\mathbf{KB}$	$=$	$\mathbf{K} + x \to \Box\Diamond x$
$\mathbf{Alt}_1$	$=$	$\mathbf{K} + \Diamond x \to \Box x$
$\mathbf{K}4$	$=$	$\mathbf{K} + \Box x \to \Box\Box x$
$\mathbf{K}5$	$=$	$\mathbf{K} + \Diamond x \to \Box\Diamond x$
$\mathbf{S}4$	$=$	$\mathbf{KT} + \Box x \to \Box\Box x$
$\mathbf{S}5$	$=$	$\mathbf{KT} + \Diamond x \to \Box\Diamond x$
$\mathbf{S}4.3$	$=$	$\mathbf{S}4 + \Box(\Box x \to y) \vee \Box(\Box y \to x)$

Important results : Let **L** be a consistent modal logic

Ladner (1977)

- If **L** $\subseteq$ **S**4 then **L** is **PSPACE**-hard

Nagle (1981)

- If **K**5 $\subseteq$ **L** then **L** is in **coNP**

Spaan (1993)

- If **S**4.3 $\subseteq$ **L** then **L** is in **coNP**

Folklore

- For all formulas $\varphi_1, \ldots, \varphi_n, \psi$, the following conditions are equivalent
  - $\varphi_1, \ldots, \varphi_n \vdash_{\mathbf{L}} \psi$
  - there exists $k_1, \ldots, k_n \in \mathbb{N}$ such that $\Box^{\leq k_1}\varphi_1 \wedge \ldots \wedge \Box^{\leq k_n}\varphi_n \rightarrow \psi$ is in **L**

Relational semantics

Frames are structures (W, R) where

- W is a nonempty set of *possible worlds* s, t, $\ldots$
- R is a binary relation of *accessibility* on W

Relational models are structures (W, R, V) where

- (W, R) is a frame
- $V \; : \; \mathbf{VAR} \cup \mathbf{PAR} \longrightarrow \wp(W)$ is a *valuation*

Relational semantics

Truth conditions in a model (W, R, V) : for all $s \in W$

$$
\begin{aligned}
s \models x &\Leftrightarrow s \in V(x) \\
s \models p &\Leftrightarrow s \in V(p) \\
s \models \bot &\Leftrightarrow \text{never} \\
s \models \neg\varphi &\Leftrightarrow s \not\models \varphi \\
s \models \varphi \vee \psi &\Leftrightarrow s \models \varphi \text{ or } s \models \psi \\
s \models \Box\varphi &\Leftrightarrow \text{for all } t \in W, \text{ if } sRt \text{ then } t \models \varphi
\end{aligned}
$$

As a result

$$s \models \Diamond\varphi \Leftrightarrow \text{there exists } t \in W \text{ such that } sRt \text{ and } t \models \varphi$$

Relational semantics

Global truth in a model : φ is *globally true* in model (W, R, V) if

- for all $s \in W$, $s \models \varphi$

Denotation : $(W, R, V) \models \varphi$

Validity in a frame : φ is *valid* in frame (W, R) if

- φ is globally true in all relational models based on (W, R)

Denotation : $(W, R) \models \varphi$

Validity in a class of frames : φ is *valid* in a class $\mathcal{C}$ of frames if

- φ is valid in all $\mathcal{C}$-frames

Denotation : $\mathcal{C} \models \varphi$

Correspondence

For all frames (W, R), the following conditions correspond

$(W, R) \models \Box x \to \Diamond x$	R is **serial**
$(W, R) \models \Box x \to x$	R is **reflexive**
$(W, R) \models x \to \Box\Diamond x$	R is **symmetric**
$(W, R) \models \Diamond x \to \Box x$	R is **deterministic**
$(W, R) \models \Box x \to \Box\Box x$	R is **transitive**
$(W, R) \models \Diamond x \to \Box\Diamond x$	R is **Euclidean**
$(W, R) \models \Box(\Box x \to y) \vee \Box(\Box y \to x)$	R is **linear**

Completeness

The following sets of formulas are equal

KD	formulas valid in all **serial frames**
KT	formulas valid in all **reflexive frames**
KB	formulas valid in all **symmetric frames**
Alt$_1$	formulas valid in all **deterministic frames**
K4	formulas valid in all **transitive frames**
K5	formulas valid in all **Euclidean frames**
S4	formulas valid in all **pre-orders**
S5	formulas valid in all **partitions**
S4.3	formulas valid in all **linear pre-orders**

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Boolean unification

Boolean language

Formulas (φ, ψ, ...) are constructed by means of

- Set **VAR** of propositional variables x, y, ...
- Set **PAR** of propositional parameters p, q, ...
- Connectives $\bot$, $\top$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$

Formal definition of the set $\mathcal{L}$ of all formulas

- $\varphi \;::=\; x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi)$

Abbreviations

- $\top$, $\wedge$, $\rightarrow$, $\leftrightarrow$ are defined as usual

Boolean unification

Proposition

Boolean unification **is unitary**

- Every **CPL**-unifiable formula has a basis of unifiers of cardinality $= 1$

Boolean elementary unification **is NP-complete**

- $\varphi(\bar{x})$ is **CPL**-unifiable $\iff$ $\varphi(\bar{x})$ is consistent

Boolean unification with parameters **is Π_2^{P}-complete**

- $\varphi(\bar{p}, \bar{x})$ is **CPL**-unifiable $\iff$ $\forall \bar{p} \exists \bar{x} \varphi(\bar{p}, \bar{x})$ is **QBF**-valid

References

- **Martin, U., Nipkow, T. :** *Boolean unification — the story so far.* Journal of Symbolic Computation **7** (1989) 275–293.
- **Baader, F. :** *On the complexity of Boolean unification.* Information Processing Letters **67** (1998) 215–220.

Boolean unification

Projective substitutions

- A substitution $\epsilon : \mathcal{L}_X \longrightarrow \mathcal{L}_X$ is **projective for a formula** φ if $\mathtt{var}(\varphi) = X$ and $\varphi \vdash_{\mathbf{CPL}} \epsilon(x) \leftrightarrow x$ for each $x \in \mathtt{var}(\varphi)$

Projective formulas

- A formula is **L-projective** if it has a projective **L**-unifier

Boolean unification

Lemma

Projective substitutions **are closed under compositions**

Proof

Suppose $\sigma : \mathcal{L}_X \longrightarrow \mathcal{L}_X$ and $\tau : \mathcal{L}_Y \longrightarrow \mathcal{L}_Y$ are such that

- $\mathtt{var}(\varphi) = X$
- $\varphi \vdash_{\mathbf{CPL}} \sigma(x) \leftrightarrow x$ for each $x \in \mathtt{var}(\varphi)$
- $\mathtt{var}(\varphi) = Y$
- $\varphi \vdash_{\mathbf{CPL}} \tau(x) \leftrightarrow x$ for each $x \in \mathtt{var}(\varphi)$

Hence

- $\varphi \vdash_{\mathbf{CPL}} \tau(\psi) \leftrightarrow \psi$ for each $\psi \in \mathcal{L}_{\mathtt{var}(\varphi)}$
- $\varphi \vdash_{\mathbf{CPL}} \tau(\sigma(x)) \leftrightarrow \sigma(x)$ for each $x \in \mathtt{var}(\varphi)$
- $\varphi \vdash_{\mathbf{CPL}} \tau(\sigma(x)) \leftrightarrow x$ for each $x \in \mathtt{var}(\varphi)$
- $\varphi \vdash_{\mathbf{CPL}} (\tau\sigma)(x) \leftrightarrow x$ for each $x \in \mathtt{var}(\varphi)$

Boolean unification

Lemma

If a substitution is projective for φ then **it is more general than any unifier of** φ

Proof

Suppose $\epsilon : \mathcal{L}_X \longrightarrow \mathcal{L}_X$ and $\tau : \mathcal{L}_X \longrightarrow \mathcal{L}_Y$ are such that

- $\mathtt{var}(\varphi) = X$
- $\varphi \vdash_{\mathbf{CPL}} \epsilon(x) \leftrightarrow x$ for each $x \in \mathtt{var}(\varphi)$
- $\tau(\varphi)$ is in **CPL**

Hence

- $\tau(\varphi) \vdash_{\mathbf{CPL}} \tau(\epsilon(x)) \leftrightarrow \tau(x)$ for each $x \in \mathtt{var}(\varphi)$
- $\tau(\epsilon(x)) \leftrightarrow \tau(x)$ is in **CPL** for each $x \in \mathtt{var}(\varphi)$
- $(\tau\epsilon)(x) \leftrightarrow \tau(x)$ is in **CPL** for each $x \in \mathtt{var}(\varphi)$
- $\tau\epsilon \simeq_{\mathbf{CPL}} \tau$
- $\epsilon \preceq_{\mathbf{CPL}} \tau$

Boolean unification

Lemma

CPL-unifiable formulas are projective

Proof: Consider a **CPL**-unifier $\sigma : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_{\mathtt{var}(\varphi)}$ of φ

- Let $\epsilon : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_{\mathtt{var}(\varphi)}$ be the substitution such that $\epsilon(x) = (\varphi \wedge x) \vee (\neg\varphi \wedge \sigma(x))$ for each $x \in \mathtt{var}(\varphi)$
- **Remarks about** ϵ
  - $\varphi \vdash_{\mathbf{CPL}} \epsilon(x) \leftrightarrow x$ for each $x \in \mathtt{var}(\varphi)$
  - $\varphi \to (\epsilon(\psi) \leftrightarrow \psi)$ is in **CPL** for each $\psi \in \mathcal{L}_{\mathtt{var}(\varphi)}$
  - $\neg\varphi \to (\epsilon(\psi) \leftrightarrow \sigma(\psi))$ is in **CPL** for each $\psi \in \mathcal{L}_{\mathtt{var}(\varphi)}$
- **Hence**
  - ϵ is a projective substitution for φ
  - $\varphi \to (\epsilon(\varphi) \leftrightarrow \varphi)$ and $\varphi \to \epsilon(\varphi)$ are in **CPL**
  - $\neg\varphi \to (\epsilon(\varphi) \leftrightarrow \sigma(\varphi))$ and $\neg\varphi \to \epsilon(\varphi)$ are in **CPL**
  - $\epsilon(\varphi)$ is in **CPL**

Proposition

Boolean unification **is unitary**

Boolean unification

The →-fragment : syntax

Formulas (φ, ψ, ...) are constructed by means of

- Set **VAR** of propositional variables x, y, ...
- Set **PAR** of propositional parameters p, q, ...
- Connectives $\bot$, $\top$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$

Formal definition of the set $\mathcal{L}_{\rightarrow}$ of all formulas

- $\varphi \ ::= \ x \mid p \mid (\varphi \rightarrow \psi)$

Abbreviations

- $\top$ is defined by $\top \ ::= \ x \rightarrow x$
- $\vee$ is defined by $(\varphi \vee \psi) \ ::= \ ((\varphi \rightarrow \psi) \rightarrow \psi)$

Boolean unification

The $\rightarrow$-fragment : some properties

Satisfiability/validity

- Every formula in $\mathcal{L}_{\rightarrow}$ is satisfiable
- The validity problem for $\mathcal{L}_{\rightarrow}$-formulas is **coNP**-complete

Axiomatization : modus ponens +

- $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))$
- $x \rightarrow (y \rightarrow x)$
- $((x \rightarrow y) \rightarrow x) \rightarrow x$

Algebraic characterization : semi-Boolean algebras $(A, \triangleright)$

- $(a \triangleright b) \triangleright a = a$
- $a \triangleright (b \triangleright c) = (b \triangleright (a \triangleright c)$
- $(a \triangleright b) \triangleright b = (b \triangleright a) \triangleright a$

Boolean unification

The $\rightarrow$-fragment : unification

For all parameter-free formulas φ in $\mathcal{L}_{\rightarrow}$

- φ is satisfiable
- φ is unifiable : a possible unifier of φ being
  - the substitution σ such that for all $x \in \mathtt{var}(\varphi)$, $\sigma(x) = \top$
- φ is projective : a possible projective unifier of φ being
  - the substitution ϵ such that for all $x \in \mathtt{var}(\varphi)$, $\epsilon(x) = \varphi \rightarrow x$

As a result

- elementary unification in the $\mathcal{L}_{\rightarrow}$-fragment of **CPL** is unitary

What is the type of

- unification with parameters in the $\mathcal{L}_{\rightarrow}$-fragment of **CPL** ?

Boolean unification

Lemma

The unifiable formula $x \to p \vee q$ is not unitary when $p \neq q$

Proof

- Suppose $\tau : \mathcal{L}_{\{x\}} \longrightarrow \mathcal{L}_Y$ is a mgu of $x \to p \vee q$
- Hence
  - $\tau(x) \to p \vee q$ is in **CPL**
  - either $p \to \tau(x)$ is in **CPL**, or $q \to \tau(x)$ is in **CPL**
- WLOG, suppose $p \to \tau(x)$ is in **CPL**
- Let $\sigma_q : \mathcal{L}_{\{x\}} \longrightarrow \mathcal{L}_\emptyset$ be such that $\sigma_q(x) = q$
- Thus, σ_q is a unifier of $x \to p \vee q$ and
  - $\tau \preceq_{\mathbf{CPL}} \sigma_q$
  - There exists $\theta_q : \mathcal{L}_Y \longrightarrow \mathcal{L}_\emptyset$ such that $\theta_q \tau \simeq_{\mathbf{CPL}} \sigma_q$
  - $(\theta_q \tau)(x) \leftrightarrow \sigma_q(x)$ is in **CPL**
  - $\theta_q(\tau(x)) \leftrightarrow q$ is in **CPL**
  - $p \to \theta_q(\tau(x))$ is in **CPL**
  - $p \to q$ is in **CPL** : a contradiction !

Boolean unification

Let φ in $\mathcal{L}_{\rightarrow}$ and $\sigma : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_Y$ be a unifier of φ
WLOG, suppose for all $x \in \mathtt{var}(\varphi)$, $\mathtt{par}(\sigma(x)) \subseteq \mathtt{par}(\varphi)$

Lemma

There exists a substitution $\epsilon : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_{\mathtt{var}(\varphi)}$ such that

- for all $x \in \mathtt{var}(\varphi)$, $\mathtt{par}(\epsilon(x)) \subseteq \mathtt{par}(\varphi)$
- for all $n \in \mathbb{N}$, $\epsilon^n \preceq_{\mathbf{CPL}} \sigma$
- for all $n \in \mathbb{N}$, if $n \geq \mathtt{Card}(\mathtt{var}(\varphi))$ then ϵ^n is a unifier of φ

Proposition

Unification with parameters in the $\mathcal{L}_{\rightarrow}$-fragment of **CPL** is finitary

Reference

- **B., P., Mojtahedi, M. :** *Unification with parameters in the implication fragment of classical propositional logic.* Logic Journal of the IGPL **30** (2022) 454–464.

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Unification types in modal logics and description logics

Propositional modal language

Formulas (φ, ψ, ...) are constructed by means of

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- Connectives $\bot$, $\top$, $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\Box$, $\Diamond$, ...

Formal definition of the set $\mathcal{L}$ of all formulas

- $\varphi ::= x \mid p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$

Abbreviations

- $\top$, $\wedge$, $\rightarrow$, $\leftrightarrow$ are defined as usual
- $\Diamond$ is defined by $\Diamond\varphi ::= \neg\Box\neg\varphi$

Unification types in modal logics and description logics

Propositional modal logic

Set **L** of formulas closed under uniform substitution and such that

1. **L** contains all tautologies
2. **L** contains the distribution axiom : $\Box(x \to y) \to (\Box x \to \Box y)$
3. **L** is closed under modus ponens : $\frac{x \quad x \to y}{y}$
4. **L** is closed under generalization : $\frac{x}{\Box x}$

Unification types in modal logics and description logics

Some computational results

Rybakov (1984, 1997)

- The unification problem and the admissibility problem in "transitive" modal logics such as $\mathbf{K}4$, $\mathbf{S}4$, ... are decidable

Chagrov (1992)

- There exists a decidable propositional modal logic with an undecidable admissibility problem

Wolter and Zakharyaschev (2008)

- The unification problem and the admissibility problem for any propositional modal logic between $\mathbf{K}_U$ and $\mathbf{K}4_U$ are undecidable

Unification types in modal logics and description logics

Some computational results

- There exists a decidable propositional modal logic with an **NP**-complete consistency problem and an undecidable admissibility problem : modal logic $\mathbf{Alt}_1 \times \mathbf{Alt}_1$

Syntax of modal logic $\mathbf{Alt}_1 \times \mathbf{Alt}_1$

Formulas

- $\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box_1\varphi \mid \Box_2\varphi$

Abbreviations

- $\top$, $\wedge$, $\rightarrow$, $\leftrightarrow$ are defined as usual
- $\Diamond_1$ is defined by $\Diamond_1\varphi \ ::= \ \neg\Box_1\neg\varphi$
- $\Diamond_2$ is defined by $\Diamond_2\varphi \ ::= \ \neg\Box_2\neg\varphi$

Unification types in modal logics and description logics

Some computational results

Axiomatization of modal logic $\mathbf{Alt}_1 \times \mathbf{Alt}_1$

- $\Box_1(x \to y) \to (\Box_1 x \to \Box_1 y)$
- $\Box_2(x \to y) \to (\Box_2 x \to \Box_2 y)$
- $\frac{x \quad x \to y}{y}$
- $\frac{x}{\Box_1 x}$
- $\frac{x}{\Box_2 x}$
- $\Diamond_1 x \to \Box_1 x$
- $\Diamond_2 x \to \Box_2 x$
- $\Box_1 \Box_2 x \leftrightarrow \Box_2 \Box_1 x$
- $\Diamond_1 \Box_2 x \to \Box_2 \Diamond_1 x$

Remark : Consistency problem in $\mathbf{Alt}_1 \times \mathbf{Alt}_1$ is **NP-complete**

- **Proof:** Small model property

Unification types in modal logics and description logics

Some computational results

Remark : Admissibility problem in $\mathbf{Alt}_1 \times \mathbf{Alt}_1$ is undecidable

Proof:

- Consider the tiling problem defined by Lutz *et al.* (2007)
  - given a finite set Δ of domino-types, binary relations V and H on Δ and subsets Δ_u, Δ_d, Δ_r and Δ_l of Δ, determine whether there exists a triple (I, J, f) where $I, J \geq 1$ and $f : \{1, \ldots, I\} \times \{1, \ldots, J\} \longrightarrow \Delta$ such that
    - for all $(i,j) \in \{1, \ldots, I-1\} \times \{1, \ldots, J\}$, $(f(i,j), f(i+1,j)) \in V$,
    - for all $(i,j) \in \{1, \ldots, I\} \times \{1, \ldots, J-1\}$, $(f(i,j), f(i,j+1)) \in H$,
    - for all $j \in \{1, \ldots, J\}$, $f(I,j) \in \Delta_u$,
    - for all $j \in \{1, \ldots, J\}$, $f(1,j) \in \Delta_d$,
    - for all $i \in \{1, \ldots, I\}$, $f(i,J) \in \Delta_r$,
    - for all $i \in \{1, \ldots, I\}$, $f(i,1) \in \Delta_l$.
- Suppose $\Delta = \{\delta_1, \ldots, \delta_a\}$ and use the propositional variables $x_1, \ldots, x_a$ and y, z ...

Proof:

- ... Construct the following 12 formulas

(ϕ_1) $\Box_2\Box_1\neg(x_b \wedge x_c)$ where $1 \leq b, c \leq a$ and $b \neq c$
(ϕ_2) $\Box_2\Box_1(x_b \rightarrow \Box_2 \bigvee\{x_c\colon (\delta_b, \delta_c) \in V\})$ where $1 \leq b \leq a$
(ϕ_3) $\Box_2\Box_1(x_b \rightarrow \Box_1 \bigvee\{x_c\colon (\delta_b, \delta_c) \in H\})$ where $1 \leq b \leq a$
(ϕ_4) $\Box_2\Box_1(y \wedge \Box_2\bot \rightarrow \bigvee\{x_b\colon \delta_b \in \Delta_u\})$
(ϕ_5) $\Box_1(y \wedge \neg z \rightarrow \Box_2(z \rightarrow \bigvee\{x_b\colon \delta_b \in \Delta_d\}))$
(ϕ_6) $\Box_2\Box_1(z \wedge \Box_1\bot \rightarrow \bigvee\{x_b\colon \delta_b \in \Delta_r\})$
(ϕ_7) $\Box_2(\neg y \wedge z \rightarrow \Box_1(y \rightarrow \bigvee\{x_b\colon \delta_b \in \Delta_l\}))$
(ϕ_8) $y \rightarrow \Box_2 y \wedge \Box_1 y$
(ϕ_9) $z \rightarrow \Box_2 z \wedge \Box_1 z$
(ϕ_{10}) $\neg y \rightarrow \Box_2\neg y$
(ϕ_{11}) $\neg z \rightarrow \Box_1\neg z$
(ϕ_{12}) $\neg(\neg y \wedge \Diamond_1 y \wedge \neg z \wedge \Diamond_2 z \wedge \Box_2\Box_1 \bigvee\{x_b :\ 1 \leq b \leq a\})$

- Show that $\frac{\phi_1\ \phi_2\ \phi_3\ \phi_4\ \phi_5\ \phi_6\ \phi_7\ \phi_8\ \phi_9\ \phi_{10}\ \phi_{11}}{\phi_{12}}$ is not admissible if and only if there exists a correct tiling of the given tiling problem

Unification types in modal logics and description logics

Some computational results

The truth is that nothing is known about the computability of the unification problem for

- **K** (elementary unification)
- **Alt**$_1$, **DAlt**$_1$, **KD**, **KT**, **KB**, **KDB**, **KTB** (unification with parameters)
- Various multimodal logics
- Various hybrid logics
- Various description logics

Unification types in modal logics and description logics

Some computational results

Remark : Elementary unification is **NP-complete** for

- any modal logic **L** containing $\Box x \to \Diamond x$ (i.e. $\mathbf{L} \supseteq \mathbf{KD}$)
- any modal logic **L** containing $x \to \Box\Diamond x$ (i.e. $\mathbf{L} \supseteq \mathbf{KB}$)
- any modal logic **L** containing $\Diamond x \to \Box\Diamond x$ (i.e. $\mathbf{L} \supseteq \mathbf{K}5$)

B. and Tinchev (2016)

- Elementary unification is **in** *PSPACE* for modal logic $\mathbf{Alt}_1$

Jeřábek (2005, 2007, 2015, 2020)

- The admissibility problem is **coNEXPTIME-complete** for intuitionistic logic and transitive modal logics like **K**4, **S**4, ...
- Unification with parameters is **coNEXPTIME-complete** for modal logic **S**5

Unification types in modal logics and description logics

Examples of unifiable formulas with their types

Examples

- In **S**5 : $\Box x \vee \Box \neg x$ is unitary
  - $\sigma(x) = \Box x$
- In **IPL** : $x \vee \neg x$ is finitary
  - $\sigma_{\top}(x) = \top$
  - $\sigma_{\bot}(x) = \bot$
- In **K**4 : $\Box x \vee \Box \neg x$ is finitary
  - $\sigma_{\top}(x) = \top$
  - $\sigma_{\bot}(x) = \bot$
- In **K** : $x \to \Box x$ is nullary
  - $\sigma_{\top}(x) = \top$
  - $\sigma_k(x) = \Box^{<k} x \wedge \Box \bot$ for each $k \in \mathbb{N}$

No known example of a modal logic with an infinitary unifiable formula

Unification types in modal logics and description logics

Modal logic K4, i.e. K $+ \Box x \to \Box\Box x$

- Syntax
  - $\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
  - $\Box^+\varphi ::= \varphi \wedge \Box\varphi$

Proposition (Rybakov 1984, 1997)

K4-unification **is decidable**

Proposition (Ghilardi 2000)

K4-unification **is finitary**

Unification types in modal logics and description logics

Ghilardi (2000) : A formula $\varphi(x_1, \ldots, x_n)$ is said to be **projective** if **there exists a substitution** σ such that

1. σ is a **K**4-unifier of φ
2. $\Box^+\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in \mathbf{K}4$ for each i such that $1 \leq i \leq n$

Wroński (1995) : A formula $\varphi(x_1, \ldots, x_n)$ is said to be **transparent** if **there exists a substitution** σ such that

1. σ is a **K**4-unifier of φ
2. for all **K**4-unifiers τ of φ, $\tau(x_i) \leftrightarrow \tau(\sigma(x_i)) \in \mathbf{K}4$ for each i such that $1 \leq i \leq n$

Unification types in modal logics and description logics

A formula $\varphi(x_1, \ldots, x_n)$ is said to be **projective** if **there exists a substitution** σ such that

1. σ is a **K**4-unifier of φ
2. $\Box^+\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in \mathbf{K}4$ for each i such that $1 \leq i \leq n$

Remark: The following statements hold:

- $\Box^+\varphi \to (\psi \leftrightarrow \sigma(\psi)) \in \mathbf{K}4$ **for each formula** $\psi(x_1, \ldots, x_n)$
- Such σ **is a most general K4-unifier** for φ
- The set of all substitutions satisfying condition 2 **is closed under compositions**

Unification types in modal logics and description logics

A formula $\varphi(x_1, \ldots, x_n)$ is said to be **projective** if **there exists a substitution** σ such that

1. σ is a **K**4-unifier of φ
2. $\Box^+\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in \mathbf{K}4$ for each i such that $1 \leq i \leq n$

For all $A \subseteq \{1, \ldots, n\}$, let θ_φ^A be the substitution defined by

- $\theta_\varphi^A(x_i) = \Box^+\varphi \wedge x_i$ if $i \notin A$
- $\theta_\varphi^A(x_i) = \Box^+\varphi \to x_i$ if $i \in A$

Remark: The substitution θ_φ^A **satisfies condition** 2

Unification types in modal logics and description logics

A formula $\varphi(x_1, \ldots, x_n)$ is said to be **projective** if **there exists a substitution** σ such that

1. σ is a **K**4-unifier of φ
2. $\Box^+\varphi \to (x_i \leftrightarrow \sigma(x_i)) \in \mathbf{K}4$ for each i such that $1 \leq i \leq n$

For all $A \subseteq \{1, \ldots, n\}$, let θ_φ^A be the substitution defined by

- $\theta_\varphi^A(x_i) = \Box^+\varphi \to x_i$ if $i \in A$
- $\theta_\varphi^A(x_i) = \Box^+\varphi \wedge x_i$ if $i \notin A$

Given **an arbitrary enumeration** $A_1, \ldots, A_{2^n}$ **of the subsets of** $\{1, \ldots, n\}$, let $\theta_\varphi = \theta_\varphi^{A_1} \circ \ldots \circ \theta_\varphi^{A_{2^n}}$

Unification types in modal logics and description logics

Proposition

For all formulas $\varphi(x_1, \ldots, x_n)$, if $d = depth(\varphi)$ and N is the number of non-d-bisimilar-equivalent relational models over $x_1, \ldots, x_n$, the following statements are equivalent :

- ${\theta_\varphi}^{2N}$ is a **K**4-unifier of φ
- φ is projective

Corollary

It is decidable to determine whether a given formula φ is projective

Unification types in modal logics and description logics

Lemma

For all formulas φ and for all substitutions σ, if $\sigma(\varphi) \in$ **K**4 then

- There exists a formula ψ, *depth*$(\psi) \leq$ *depth*(φ), such that
  - ψ is projective
  - σ is a **K**4-unifier of ψ
  - $\Box^{+}\psi \rightarrow \varphi \in$ **K**4

Proposition

K4-unification **is finitary,**

- For all formulas $\varphi(x_1, \ldots, x_n)$, the cardinality of a basis of **K**4-unifiers **is finite**

Reference

- **Ghilardi, S. :** *Best solving modal equations.* Annals of Pure and Applied Logic **102** (2000) 183–198.

Unification types in modal logics and description logics

Intuitionistic propositional logic — IPL

Ghilardi (1999) :

- for every **IPL**-unifiable formula φ, one can find a finite number of projective formula $\psi_1, \ldots, \psi_n$ such that (i) for all $k = 1 \ldots n$, $\psi_k \to \varphi$ is in **IPL** and (ii) every **IPL**-unifier for φ is also an **IPL**-unifier for one of the $\psi_1, \ldots, \psi_n$

Logic of Gödel and Dummett — LC

LC is **IPL** $+ (x \to y) \vee (y \to x)$

Wroński (2008) :

- In all extensions of **LC**, unifiable formulas have projective unifiers
- An intermediate logic **L** in which all unifiable formulas have projective unifiers must contain **LC**

Unification types in modal logics and description logics

Extensions of **S**4.3

S4.3 is **S**4 $+ \Box(\Box x \to y) \vee \Box(\Box y \to x)$

Dzik and Wojtylak (2011) :

- In all extensions of **S**4.3, unifiable formulas have projective unifiers
- Extensions of **S**4 in which all unifiable formulas have projective unifiers must contain **S**4.3

Extensions of **K**4**D**1

K4**D**1 is **K**4 $+ \Box(\Box x \to y) \vee \Box(\Box y \to x)$

Kost (2018) :

- In all extensions of **K**4**D**1, unifiable formulas have projective unifiers
- Extensions of **K**4 in which all unifiable formulas have projective unifiers must contain **K**4**D**1

Unification types in modal logics and description logics

Modal logic S5, i.e. KT $+ \Diamond x \to \Box\Diamond x$

- Syntax
  - $\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$

Proposition

S5-unification **is decidable**

Proposition

S5-unification **is unitary**

Unification types in modal logics and description logics

Lemma

S5-unifiable formulas **are S5-projective**

Proof: Consider an **S**5-unifier σ of φ

- Let ϵ be the substitution such that
  $\epsilon(x) = (\Box\varphi \wedge x) \vee (\neg\Box\varphi \wedge \sigma(x))$
- **Fact :** ϵ is a projective unifier of φ

Proposition

S5 unification **is unitary :** every unifiable formula has a mgu

Remark about ϵ

- If σ is atom-free then ϵ can be defined by
  - $\epsilon(x) = \Box\varphi \wedge x$ when $\sigma(x) = \bot$
  - $\epsilon(x) = \Box\varphi \rightarrow x$ when $\sigma(x) = \top$

Unification types in modal logics and description logics

Remark

- The proofs that **CPL** and **S**5 are unitary are based on the fact that every unifiable formula is projective in these logics

It is true that

- **if every L-unifiable formula has a projective unifier then L-unification is unitary**

However

- **S**4.2**Grz-unification is unitary** (Ghilardi 2000)
- **some S**4.2**Grz-unifiable formulas are not projective** (Dzik 2006)

Unification types in modal logics and description logics

Modal logic K

- Syntax
  - $\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$
- Abbreviations
  - $\Diamond\varphi ::= \neg\Box\neg\varphi$
  - $\Box^{<n}\varphi ::= \Box^{0}\varphi \wedge \ldots \wedge \Box^{n-1}\varphi$ for each $n \in \mathbb{N}$

Open question

Is K-unification decidable ?

Remark

K-unification **is not unitary** since

- $\sigma_{\top}(x) = \top$ and $\sigma_{\bot}(x) = \bot$ constitute a basis of unifiers in **K** of the formula $\Box x \vee \Box\neg x$

Unification types in modal logics and description logics

Our aim

Demonstrate that **K**-unification **is nullary** by studying the **K**-unifiers of

- $x \to \Box x$

Consider the following substitutions

- $\sigma_n(x) = \Box^{<n} x \wedge \Box^n \bot$ for each $n \in \mathbb{N}$
- $\sigma_\top(x) = \top$

Lemma

- σ_n **is a K-unifier** of $x \to \Box x$ for each $n \in \mathbb{N}$
- $\sigma_\top$ **is a K-unifier** of $x \to \Box x$

Unification types in modal logics and description logics

Our aim

Demonstrate that **K**-unification **is nullary** by studying the **K**-unifiers of

- $x \to \Box x$

Consider the following substitutions

- $\sigma_n(x) = \Box^{<n} x \wedge \Box^n \bot$ for each $n \in \mathbb{N}$
- $\sigma_\top(x) = \top$

Lemma

For all **K**-unifiers σ of $x \to \Box x$ and for all $n \in \mathbb{N}$, $\sigma \preceq_K \sigma_n$ if and only if $\sigma(x) \to \Box^n \bot \in \mathbf{K}$

Lemma

For all substitutions σ, $\sigma \preceq_K \sigma_\top$ if and only if $\sigma(x) \in \mathbf{K}$

Unification types in modal logics and description logics

Proposition

For all formulas φ and for all $n \in \mathbb{N}$, if $depth(\varphi) \leq n$ then

- If $\varphi \to \Box\varphi \in \mathbf{K}$ then either $\varphi \to \Box^n\bot \in \mathbf{K}$, or $\varphi \in \mathbf{K}$

Corollary

The following substitutions form a complete set of **K**-unifiers for the formula $x \to \Box x$

- $\sigma_n(x) = \Box^{<n}x \wedge \Box^n\bot$ for each $n \in \mathbb{N}$
- $\sigma_\top(x) = \top$

Corollary

K-unification **is nullary**

Reference

- **Jeřábek, E. :** *Blending margins: the modal logic* **K** *has nullary unification type.* Journal of Logic and Computation **25** (2015) 1231–1240.

Unification types in modal logics and description logics

Directed unification

L has directed unification if for all **L**-unifiable formulas φ and for all **L**-unifiers $\sigma : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_Y$ and $\tau : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_Z$ of φ, there exists an **L**-unifier $\theta : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_T$ of φ such that

- $\theta \preceq_{\mathbf{L}} \sigma$
- $\theta \preceq_{\mathbf{L}} \tau$

Lemma

If **L** has directed unification then either **L** is unitary, or **L** is nullary

Unification types in modal logics and description logics

Extensions of $\mathbf{K}4 = \mathbf{K} + \Box x \to \Box\Box x$

- Define the abbreviations
  - $\Box^+\varphi := (\Box\varphi \wedge \varphi)$
  - $\Diamond^+\varphi := (\Diamond\varphi \vee \varphi)$
- $\mathbf{K}4.2^+$ is $\mathbf{K}4 + \Diamond^+\Box^+\varphi \to \Box^+\Diamond^+\varphi$
- An extension $\mathbf{L}$ of $\mathbf{K}4$ has **directed unification** if and only if $\mathbf{K}4.2^+ \subseteq \mathbf{L}$

References

- **Ghilardi, S., Sacchetti, L. :** *Filtering unification and most general unifiers in modal logic.* Journal of Symbolic Logic **69** (2004) 879–906.
- **Jeřábek, E. :** *Logics with directed unification.* In : *Algebra and Coalgebra meet Proof Theory,* Utrecht, Netherlands (2013).

Unification types in modal logics and description logics

Extensions of $\mathbf{K}5 = \mathbf{K} + \Diamond x \rightarrow \Box\Diamond x$

Remark : Every extension $\mathbf{L}$ of $\mathbf{K}5$ has directed unification

Proof : Consider an $\mathbf{L}$-unifiable formula φ

- Let $\sigma : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_Y$ and $\tau : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_Z$ be $\mathbf{L}$-unifiers of φ and t be a new propositional variable
- Let $\theta : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_{Y \cup Z \cup \{t\}}$ be the substitution defined for all $x \in \mathtt{var}(\varphi)$ by
  - $\theta(x) = ((\Box\Box t \wedge (t \vee \Diamond\top)) \wedge \sigma(x)) \vee ((\Diamond\Diamond\neg t \vee (\neg t \wedge \Box\bot)) \wedge \tau(x))$
- One can prove that
  - $\theta \preceq_{\mathbf{L}} \sigma$
  - $\theta \preceq_{\mathbf{L}} \tau$
  - $\theta(\varphi)$ is in $\mathbf{L}$

Reference

- **Alizadeh, M., Ardeshir, M., B., P., Mojtahedi, M. :** *Unification types in Euclidean modal logics.* Logic Journal of the IGPL (to appear).

Unification types in modal logics and description logics

Description language $\mathcal{FL}_0$

The set of all concepts is defined by

- $C ::= X \mid A \mid \top \mid (C \sqcap D) \mid \forall R.C$

Two concept descriptions C, D are **equivalent** ($C \equiv D$) if

- $C \leftrightarrow D$ is valid in the class of all frames

Proposition Equivalence of $\mathcal{FL}_0$-concepts **can be decided in polynomial time**

Reference

- **Levesque, H., Brachman, R. :** *Expressiveness and tractability in knowledge representation and reasoning.* Computational Intelligence **3** (1987) 78–93.

Unification types in modal logics and description logics

Description language $\mathcal{FL}_0$

The substitution σ **unifies** the concept descriptions C and D if

- $\sigma(C) \equiv \sigma(D)$

C and D are $\mathcal{FL}_0$-**unifiable** if they have a unifier

Example The substitution σ defined by

- $\sigma(X) = A \sqcap \forall S.A$
- $\sigma(Y) = \forall R.A$

is a unifier of the $\mathcal{FL}_0$-concept descriptions

- $C = \forall R.\forall R.A \sqcap \forall R.X$
- $D = Y \sqcap \forall R.Y \sqcap \forall R.\forall S.A$

Unification types in modal logics and description logics

Description language $\mathcal{FL}_0$

Proposition

- Unification in idempotent Abelian monoids with homomorphism is nullary
- $\mathcal{FL}_0$ is nullary — try to unify $\forall R.X \sqcap \forall R.Y$ and $Y \sqcap \forall R.\forall R.Z$

Proposition

- Solvability of unification problems in $\mathcal{FL}_0$ can be decided in deterministic exponential time

References

- **Baader, F. :** *Unification in commutative theories.* Journal of Symbolic Computation **8** (1989) 479–497.
- **Baader, F., Narendran, P. :** *Unification of concept terms in description logics.* Journal of Symbolic Computation **31** (2001) 277–305.

Unification types in modal logics and description logics

Description language $\mathcal{EL}$

Syntax of the **description language** $\mathcal{EL}$

- $C ::= X \mid A \mid \top \mid (C \sqcap D) \mid \exists R.C$

Two concept descriptions C, D are **equivalent** ($C \equiv D$) if

- $C \leftrightarrow D$ is valid in the class of all frames

Proposition Equivalence of $\mathcal{EL}$-concept descriptions **can be decided in polynomial time**

Reference

- **Baader, F., Molitor, R., Tobies, S. :** *Tractable and decidable fragments of conceptual graphs.* In Tepfenhart, W., Cyre, W. (editors) : Conceptual Structures: Standards and Practices. Springer (1999) 480–493.

Unification types in modal logics and description logics

Description language $\mathcal{EL}$

The substitution σ **unifies** the concept descriptions C and D if

- $\sigma(C) \equiv \sigma(D)$

C and D are **$\mathcal{EL}$-unifiable** if they have a unifier

Example The substitution σ defined by

- $\sigma(X) = \top$
- $\sigma(Y) = Y$

is a unifier of the $\mathcal{EL}$-concept descriptions

- $C = X \sqcap \exists R.Y$
- $D = \exists R.Y$

Unification types in modal logics and description logics

Description language $\mathcal{EL}$

Proposition

- Unification in $\mathcal{EL}$ **is NP-complete**
- $\mathcal{EL}$ is nullary — try to unify $X \sqcap \exists R.Y$ and $\exists R.Y$

Proposition

- Unification in $\mathcal{EL}^{-\top}$ **is PSPACE-complete**

References

- **Baader, F., Binh, N., Borgwardt, S., Morawska, B. :** *Unification in the description logic $\mathcal{EL}$ without the top concept.* In Bjørner, N., Sofronie-Stokkermans, V. (editors) : Automated Deduction — CADE 23. Springer (2011) 70–84.
- **Baader, F., Morawska, B. :** *Unification in the description logic $\mathcal{EL}$.* In Treinen, R. (editor) : Rewriting Techniques and Applications. Springer (2009) 350–364.

Outline

Contents

- Definitions
- Boolean unification
- Unification types in modal logics and description logics
- **Recent advances**

Recent advances

Restricted unification in the description logics $\mathcal{FL}_0$ and $\mathcal{EL}$

Proposition

- $\mathcal{FL}_0$ with restriction on role depth is finitary
- $\mathcal{EL}$ with restriction on role depth is nullary

References

- **Baader, F., Fernández Gil, O., Rostamigiv, M. :** *Restricted unification in the DL $\mathcal{FL}_0$.* In: Frontiers of Combining Systems. Springer (2021) 81–97.
- **Baader, F., Rostamigiv, M. :** *Restricted unification in the DL $\mathcal{EL}$.* In: Proceedings of the 34th International Workshop on Description Logics. CEUR (2021) paper 4.

Recent advances

Elementary unification in $\mathbf{K} + \Box^n\bot$ and $\mathbf{Alt}_1 + \Box^n\bot$ for $n \geq 2$

Proposition : For all $n \geq 2$

- $\mathbf{K} + \Box^n\bot$ is not unitary for elementary unification
- $\mathbf{Alt}_1 + \Box^n\bot$ is either unitary, or nullary for elementary unification

Proof : Case when $n = 2$

- For $\mathbf{K} + \Box^2\bot$, consider the formula $\Box x \vee \Box\neg x$ and show that the substitutions $\sigma_\top(x) = \Diamond\top \wedge x$ and $\sigma_\bot(x) = \Box\bot \vee x$ constitute a basis of unifiers
- For $\mathbf{Alt}_1 + \Box^2\bot$, show that $\mathbf{Alt}_1 + \Box^n\bot$ is directed for elementary unification

Recent advances

Elementary unification in $\mathbf{K} + \Box^n\bot$ and $\mathbf{Alt}_1 + \Box^n\bot$ for $n \geq 2$

Let φ be unifiable and $\sigma : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_Y$ be a unifier of φ

- **Lemma :** There exists an unifier $\tau : \mathcal{L}_{\mathtt{var}(\varphi)} \longrightarrow \mathcal{L}_{\mathtt{var}(\varphi)}$ of φ such that $\tau \preceq \sigma$
- **Proposition :** Elementary unification in $\mathbf{K} + \Box^n\bot$ and $\mathbf{Alt}_1 + \Box^n\bot$ is either unitary, or finitary
- **Corollary :** Elementary unification in $\mathbf{K} + \Box^n\bot$ is finitary and elementary unification in $\mathbf{Alt}_1 + \Box^n\bot$ is unitary

References

- **B., P., Gencer, Ç. Rostamigiv, M., Tinchev, T. :** *About the unification type of* $\mathbf{K} + \Box\Box\bot$*.* Annals of Mathematics and Artificial Intelligence **90** (2022) 481–497.
- **B., P., Gencer, Ç. Rostamigiv, M., Tinchev, T. :** *Remarks about the unification types of some locally tabular normal modal logics.* Logic Journal of the IGPL (to appear).

Recent advances

Extensions of $\mathbf{K}5 \;=\; \mathbf{K} + \Diamond x \to \Box\Diamond x$

Proposition (elementary unification)

- Extensions of $\mathbf{K}45 = \mathbf{K}5 + \Box x \to \Box\Box x$ are projective
- $\mathbf{K}5$ and $\mathbf{KD}5$ are unitary

Open question

Are all extensions of $\mathbf{K}5$ unitary for elementary unification ?

References

- **Alizadeh, M., Ardeshir, M., B., P., Mojtahedi, M. :** *Unification types in Euclidean modal logics.* Logic Journal of the IGPL (to appear).
- **Kost, S. :** *Projective unification in transitive modal logics.* Logic Journal of the IGPL **26** (2018) 548–566.

Recent advances

$$\mathbf{KD} = \mathbf{K} + \Diamond\top$$

Proposition

KD is nullary for unification with parameters

- $(x \to p) \wedge (x \to \Box(p \to x))$

Open questions

Type of **KD** for elementary unification ?
Decidability of unification with parameters in **KD** ?

Reference

- **B., P., Gencer, Ç. : KD** *is nullary.* Journal of Applied Non-Classical Logics **27** (2018) 196–205.

Recent advances

$\mathbf{KT} \; = \; \mathbf{K} + \Box x \to x$

Proposition

KT is nullary for unification with parameters

- $(x \to p) \wedge (x \to \Box(q \to y)) \wedge (y \to q) \wedge (y \to \Box(p \to x))$

Open questions

Type of **KT** for elementary unification ?
Decidability of unification with parameters in **KT** ?

Reference

- **B., P. :** *Remarks about the unification type of several non-symmetric non-transitive modal logics.* Logic Journal of the IGPL **27** (2019) 639–658.

Recent advances

$$\mathbf{KB} = \mathbf{K} + x \to \Box\Diamond x$$

Proposition

KB is nullary for unification with parameters

- $x \to (\neg p \wedge \neg q \to \Box(p \wedge \neg q \to \Box(\neg p \wedge q \to \Box(\neg p \wedge \neg q \to x))))$

Open questions

Type of **KB** for elementary unification ?
Decidability of unification with parameters in **KB** ?

Reference

- **B., P., Gencer, Ç. :** *About the unification type of modal logics between* **KB** *and* **KTB**. Studia Logica **108** (2020) 941–966.

Recent advances

$$\mathbf{Alt}_1 \;=\; \mathbf{K} + \Diamond x \to \Box x$$

Proposition

- $\mathbf{Alt}_1$ is nullary — try to unify $x \to \Box x$
- The elementary unification problem (without parameters) in $\mathbf{Alt}_1$ is decidable (in **PSPACE**)

Open question

Decidability of unification with parameters in $\mathbf{Alt}_1$?

Reference

- **B., P., Tinchev, T. :** *Unification in modal logic* $\mathbf{Alt}_1$. In Beklemishev, L., Demri, S., Máté, A. (editors) : Advances in Modal Logic. Volume 11. College Publications (2016) 117–134.

Conclusion

Some open problems

Decidability of

- elementary unification in modal logic **K** ?
- unification with parameters in modal logic **KB** ? in modal logics **KD**, **KDB** ? in modal logics **KT**, **KTB** ? in modal logic $\mathbf{Alt}_1$?
- unification in the implicative fragment of modal logics ?
- unification in the positive fragment of modal logics ?

Exact complexity of

- unification in $\mathbf{Alt}_1$, **K**4, **S**4, . . .

Type of

- **KB**, **KD**, **KDB**, **KT**, **KTB** for elementary unification ?
- fusions of modal logics ? Products of modal logics ?
- non-transitive extensions of **K**5 and other locally tabular modal logics ?
- unification in the implicative fragment of modal logics ?
- unification in the positive fragment of modal logics ?

Thank you !

Bibliography

- Alizadeh, M., M. Ardeshir,, P. Balbiani, and M. Mojtahedi, 'Unification types in Euclidean modal logics', *Logic Journal of the IGPL* (to appear).
- Anantharaman, S., P. Narendran, and M. Rusinowitch, 'Unification modulo **ACUI** plus distributivity axioms', *Journal of Automated Reasoning* 33:1–28, 2004.
- Baader, F., 'Unification in commutative theories', *Journal of Symbolic Computation* **8** (1989) 479–497.
- Baader, F., 'On the complexity of Boolean unification', *Information Processing Letters* **67** (1998) 215–220.
- Baader, F., N. Binh, S. Borgwardt, and B. Morawska, 'Unification in the description logic $\mathcal{EL}$ without the top concept', In: *Automated Deduction — CADE 23,* Springer (2011) 70–84.

Bibliography

- BAADER, F., O. FERNÁNDEZ GIL and M. ROSTAMIGIV, 'Restricted unification in the DL $\mathcal{FL}_0$', In: *Frontiers of Combining Systems,* Springer (2021) 81–97.
- BAADER, F., and S. GHILARDI, 'Unification in modal and description logics', *Logic Journal of the IGPL* **19** (2011) 705–730.
- BAADER, F., and R. KÜSTERS, 'Unification in a description logic with transitive closure of roles', In: *Logic for Programming and Automated Reasoning,* Springer (2001) 217–232.
- BAADER, F., and R. KÜSTERS, 'Nonstandard inferences in description logics: the story so far', In: *Mathematical Problems from Applied Logic I,* Springer (2006) 1–76.
- BAADER, F., R. MOLITOR, and S. TOBIES, 'Tractable and decidable fragments of conceptual graphs', In: *Conceptual Structures: Standards and Practices,* Springer (1999) 480–493.

Bibliography

- Baader, F., and B. Morawska, 'Unification in the description logic $\mathcal{EL}$', In: *Rewriting Techniques and Applications*, Springer (2009) 350–364.
- Baader, F., and B. Morawska, '*SAT* encoding of unification in $\mathcal{EL}$', In: *Logic for Programming, Artificial Intelligence, and Reasoning*, Springer (2010) 97–111.
- Baader, F., and P. Narendran, 'Unification of concept terms in description logics', *Journal of Symbolic Computation* **31** (2001) 277–305.
- Baader, F., and M. Rostamigiv, 'Restricted unification in the DL $\mathcal{EL}$', In: *Proceedings of the 34th International Workshop on Description Logics*, CEUR (2021) paper 4.
- Baader, F., and W. Snyder, 'Unification theory', In: *Handbook of Automated Reasoning*, Elsevier (2001) 439–526.

Bibliography

- Babenyshev, S., and V. Rybakov, 'Linear temporal logic **LTL**: basis for admissible rules', *Journal of and Computation* **21** (2010) 157–177.
- Babenyshev, S., and V. Rybakov, 'Unification in linear temporal logic **LTL**', *Annals of Pure and Applied Logic* **162** (2011) 991–1000.
- Babenyshev, S., V. Rybakov, R. Schmidt, and D. Tishkovsky, 'A tableau method for checking rule admissibility in **S**4', *Electronic Notes in Theoretical Computer Science* **262** (2010) 17–32.
- Balbiani, P., 'Remarks about the unification type of several non-symmetric non-transitive modal logics', *Logic Journal of the IGPL* **27** (2019) 639–658.
- Balbiani, P., and Ç. Gencer, '**KD** is nullary', *Journal of Applied Non-Classical Logics* **27** (2017) 196–205.

Bibliography

- Balbiani, P., and Ç. Gencer, 'Unification in epistemic logics', *Journal of Applied Non-Classical Logics* **27** (2017) 91–105.
- Balbiani, P., and Ç. Gencer, 'About the unification type of modal logics between **KB** and **KTB**', *Studia Logica* **108** (2020) 941–966.
- Balbiani, P., Ç. Gencer, M. Rostamigiv, and T. Tinchev, 'About the unification type of $\mathbf{K} + \Box\Box\bot$', *Annals of Mathematics and Artificial Intelligence* **90** (2022) 481–497.
- Balbiani, P., Ç. Gencer, M. Rostamigiv, and T. Tinchev, 'Remarks about the unification types of some locally tabular normal modal logics', *Logic Journal of the IGPL* (to appear).
- Balbiani, P., and M. Mojtahedi, 'Unification with parameters in the implication fragment of Classical Propositional Logic', *Logic Journal of the IGPL* **30** (2022) 454–464.

Bibliography

- Balbiani, P., and T. Tinchev, 'Unification in modal logic **Alt**$_1$', In: *Advances in Modal Logic,* College Publications (2016) 117–134.
- Balbiani, P., and T. Tinchev, 'Elementary unification in modal logic **KD**45', *Journal of Applied Logics* **5** (2018) 301–317.
- Bezhanishvili, N., and D. De Jongh, *Extendible formulas in two variables in intuitionistic logic, Studia Logica* **100** (2012) 61–89.
- Blackburn, P., M. de Rijke, and Y. Venema, *Modal Logic,* Cambridge University Press (2001).
- Büttner, W., and H. Simonis, 'Embedding Boolean expressions into logic programming', *Journal of Symbolic Computation* **4** (1987) 191–205.

Bibliography

- CHAGROV, A., ‘Decidable modal logic with undecidable admissibility problem’, *Algebra and Logic* **31** (1992) 83–93.
- CHAGROV, A., and M. ZAKHARYASCHEV, *Modal Logic*, Oxford University Press (1997).
- CINTULA, P., and G. METCALFE, ‘Admissible rules in the implication-negation fragment of intuitionistic logic’, *Annals of Pure and Applied Logic* **162** (2010) 162–171.
- DZIK, W., ‘Unitary unification of **S**5 modal logics and its extensions’, *Bulletin of the Section of Logic* **32** (2003) 19–26.
- DZIK, W., ‘Transparent unifiers in modal logics with self-conjugate operators’, *Bulletin of the Section of Logic* **35** (2006) 73–83.

Bibliography

- Dzik, W., *Unification Types in Logic,* Wydawnicto Uniwersytetu Slaskiego (2007).
- Dzik, W., 'Unification and slices in intermediate and in some modal logics', In: *Topology, Algebra and Categories in Logic,* Amsterdam, Netherlands (2009).
- Dzik, W., 'Remarks on projective unifiers', *Bulletin of the Section of Logic* **40** (2011) 37–46.
- Dzik, W., S., Kost, and P. Wojtylak, 'Finitary unification in locally tabular modal logics characterized', *Annals of Pure and Applied Logic* (to appear).
- Dzik, W., and P. Wojtylak, 'Projective unification in modal logic', *Logic Journal of the IGPL* **20** (2012) 121–153.

Bibliography

- Dzik, W., and P. Wojtylak, 'Modal consequence relations extending **S**4.3: an application of projective unification', *Notre Dame Journal of Formal Logic* **57** (2013) 523–549.
- Fernández Gil, O., *Hybrid Unification in the Description Logic $\mathcal{EL}$*, Master Thesis of Technische Universität Dresden (2012).
- Gencer, Ç., and D. de Jongh, 'Unifiability in extensions of **K**4', *Logic Journal of the IGPL* **17** (2009) 159–172.
- Ghilardi, S., 'Unification through projectivity', *Journal of Logic and Computation* **7** (1997) 733–752.
- Ghilardi, S., 'Unification in intuitionistic logic', *Journal of Symbolic Logic* **64** (1999) 859–880.

Bibliography

- GHILARDI, S., ‘Best solving modal equations’, *Annals of Pure and Applied Logic* **102** (2000) 183–198.
- GHILARDI, S., and L. SACCHETTI, ‘Filtering unification and most general unifiers in modal logic’, *Journal of Symbolic Logic* **69** (2004) 879–906.
- IEMHOFF, R., ‘On the admissible rules of intuitionistic propositional logic’, *Journal of Symbolic Computation* **66** (2001) 281–294.
- IEMHOFF, R., ‘A syntactic approach to unification in transitive reflexive modal logics’, *Notre Dame Journal of Formal Logic* **57** (2016) 233–247.
- IEMHOFF, R., and G. METCALFE, ‘Proof theory for admissible rules’, *Annals of Pure and Applied Logic* **159** (2009) 171–186.

Bibliography

- JEŘÁBEK, E., ‘Complexity of admissible rules’, *Archive for Mathematical Logic* **46** (2007) 73–92.
- JEŘÁBEK, E., ‘Logics with directed unification’, In: *Algebra and Coalgebra meet Proof Theory*, Utrecht, Netherlands (2013).
- JEŘÁBEK, E., ‘Blending margins: the modal logic **K** has nullary unification type’, *Journal of Logic and Computation* **25** (2015) 1231–1240.
- KNUTH, D., and P. BENDIX, ’Simple word problems in universal algebras’, In: *Computational Problems in Abstract Algebra*, Pergamon Press (1970) 263–297.
- KOST, S., ‘Projective unification in transitive modal logics’, *Logic Journal of the IGPL* **26** (2018) 548–566.

Bibliography

- Levesque, H., and R. Brachman, 'Expressiveness and tractability in knowledge representation and reasoning', *Computational Intelligence* **3** (1987) 78–93.
- Löwenheim, L., 'Über das Auflösungsproblem im logischen Klassenkalkül', *Sitzungsberichte der Berliner mathematischen Gesellschaft* **7** (1908) 89–94.
- Lutz, C., D. Walther, and F. Wolter, 'Conservative extensions in expressive description logics', In: *IJCAI'07: Proceedings of the 20th international joint conference on Artifical intelligence,* Morgan Kaufmann (2007) 453–458.
- Martin, U., T. Nipkow, 'Unification in Boolean rings', *Journal of Automated Reasoning* **4** (1988) 381–396.
- Martin, U., and T. Nipkow, 'Boolean unification — the story so far', *Journal of Symbolic Computation* **7** (1989) 275–293.
- Prucnal, T., 'On the structural completeness of some pure implicational propositional calculi', *Studia Logica* **30** (1972) 45–50.

Bibliography

- Robinson, J., 'A machine oriented logic based on the resolution principle', *Journal of the ACM* **12** (1965) 23–41.
- Rostamigiv, M., *About the Type of Modal Logics for the Unification Problem*, Doctoral thesis of the University of Toulouse 3 (2020).
- Rudeanu, S., *Boolean Functions and Equations*, Elsevier (1974).
- Rybakov, V., 'Admissible rules for pretable modal logics', *Algebra and Logic* **20** (1981) 440–464
- Rybakov, V., 'A criterion for admissibility of rules in the model system $S4$ and the intuitionistic logic. *Algebra and Logic* **23** (1984) 369–384.

Bibliography

- RYBAKOV, V., 'Bases of admissible rules of the logics *S*4 and *Int*', *Algebra and Logic* **24** (1985) 55–68.
- RYBAKOV, V., *Admissibility of Logical Inference Rules,* Elsevier (1997).
- RYBAKOV, V., 'Construction of an explicit basis for rules admissible in modal system **S**4', *Mathematical Logic Quarterly* **47** (2001) 441–446.
- RYBAKOV, V., 'Linear temporal logic with until and next, logical consecutions', *Annals of Pure and Applied Logic* **155** (2008) 32–45.
- RYBAKOV, V., M. TERZILER, and Ç. GENCER, 'An essay on unification and inference rules for modal logics', *Bulletin of the Section of Logic* **28** (1999) 145–157.

Bibliography

- RYBAKOV, V., M. TERZILER, and Ç. GENCER, 'An essay on unification and inference rules for modal logics', *Bulletin of the Section of Logic* **28** (1999) 145–157.
- SIEKMANN, J., 'Unification theory', *Journal of Symbolic Computation* **7** (1989) 207–274.
- SOFRONIE-STOKKERMANS, V., 'Locality and subsumption testing in $\mathcal{EL}$ and some of its extensions', In: *Advances in Modal Logic. Volume 7,* College Publications (2008) 315–339.
- WOLTER, F., and M. ZAKHARYASCHEV, 'Undecidability of the unification and admissibility problems for modal and description logics', *ACM Transactions on Computational Logic* 9:25:1–25:20, 2008.
- WROŃSKI, A., 'Transparent unification problem', *Reports on Mathematical Logic* **29** (1995) 105–107.

Bibliography

- WROŃSKI, A., 'Unitary unification for equivalential algebras and other structures related to logic', In: *11th International Congress of Logic, Methodology and Philosophy of Science,* Cracow, Poland (1999).
- WROŃSKI, A., 'Transparent verifiers in intermediate logics', In: *54th Conference in History of Mathematics,* Cracow, Poland (2008).