# Two results about dense inhomogeneous random graphs

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Based on

- Doležal, H., Máthé: Cliques in dense inhomogeneous random graphs

  Random Structures and Algorithms 2017
- H., Viswanathan: Connectivity of inhomogeneous random graphs II

  arXiv: 2305.03607

# Erdős–Rényi random graph **G**($n$,$p$)

- Definition of **G**($n$,$p$): $n \in \mathbb{N}$, $p \in [0,1]$; vertex set $\{1,\ldots,n\}$,
  make each pair of vertices an edge with probability $p$.
- Introduced by Gilbert 1958, Erdős–Rényi 1959.
- Usually, exciting things happen when $p=p(n)$ tends to 0
  - Giant component: $p$ = const / $n$.
  - Hamiltonicity: $p$ = const * log $n$ / $n$.

# Erdős–Rényi random graph $\boldsymbol{G}(n,p)$

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- Theorem [Gilbert 1958, Erdős–Rényi 1959]: Let $\varepsilon>0$ be fixed.
  - If $p(n)>(1+\varepsilon)\ln n / n$, then $\boldsymbol{G}(n,p)$ is asymptotically almost surely connected.
  - If $p(n)<(1-\varepsilon)\ln n / n$, then $\boldsymbol{G}(n,p)$ a. a. s. contains an isolated vertex. In particular, it is disconnected.
- Theorem [Grimmett-McDiarmid 1975, Matula 1976]: Let $\varepsilon>0$ and $p\in(0,1)$ be fixed. The clique number a.a.s. satisfies

$$\omega(\boldsymbol{G}(n,p))=(2 \pm \varepsilon)\ln n / \ln(1/p).$$

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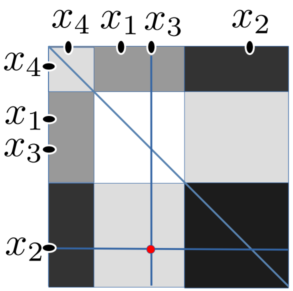
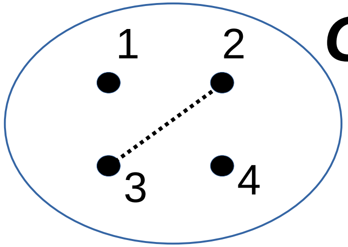
Lower bound for diagonal Ramsey numbers $R(k,k)>(\sqrt{2})^k$.

# Graphon based random graphs

- Definition of graphon: $W$:$[0,1]^2 \rightarrow [0,1]$ measurable, symmetric $W(x,y)=W(y,x)$
- Definition of $\boldsymbol{G}(n,W)$: Vertex set $\{1,\ldots,n\}$
  - Generate $x_1, x_2, \ldots, x_n \in [0,1]$ at random.
  - For each pair $\{i,j\}$, insert it as an edge with probability $W(x_i, x_j)=W(x_j, x_i)$.
- Introduced by Lovász and Szegedy 2006

  ”Every graphon can be approximated by finite graphs.”




$\boldsymbol{G}(n=4,W)$

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Stochastic block model

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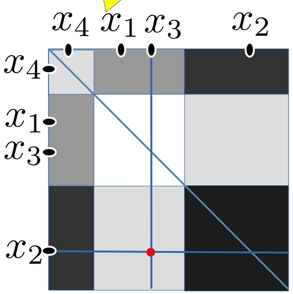


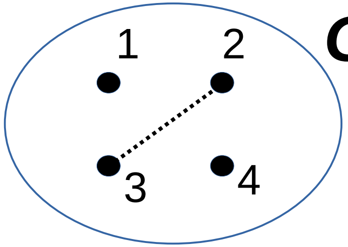


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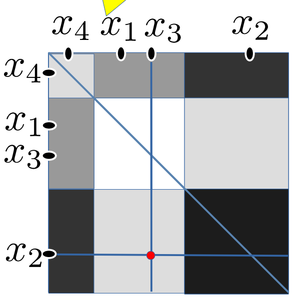
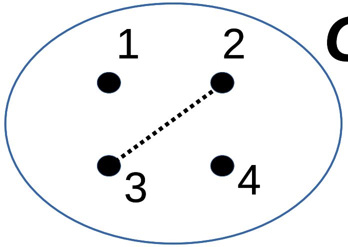
- …ász and…
  - Every graphon can be ap…

Stochastic block model

$\mathbf{G}(n,W)$ is typically dense (and thus boring???)

- $W\equiv 0$, or
- $W\neq 0$ and then a.a.s. $e(\mathbf{G}(n,W))=(0.5\pm\varepsilon)\|W\|_1\, n^2$




$\mathbf{G}(n=4,W)$

# Connectivity of $\mathbf{G}(n,W)$

- Theorem [Gilbert 1958, Erdős–Rényi 1959]: Let ε>0 be fixed.
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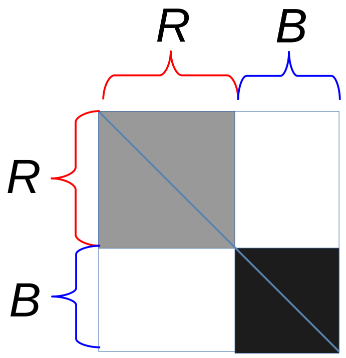
# Connectivity of **G**($n$,$W$)

- First Obstacle: Disconnectedness of $W$




Definition: $W$ is *disconnected* if there exists a partition $[0,1]=R\cup B$ into two sets of positive measure such that $W$ is zero on $R\times B$.

# Connectivity of $\boldsymbol{G}(n,W)$

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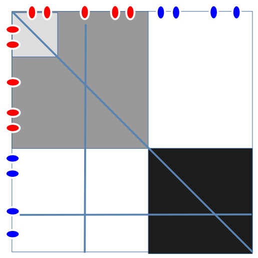
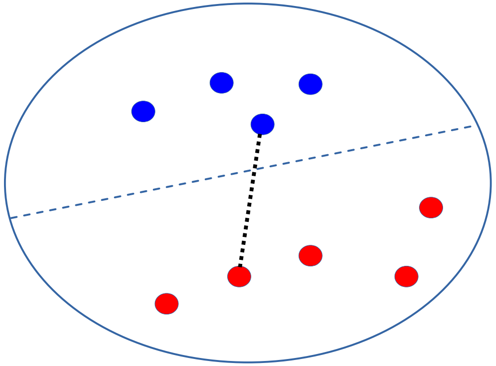
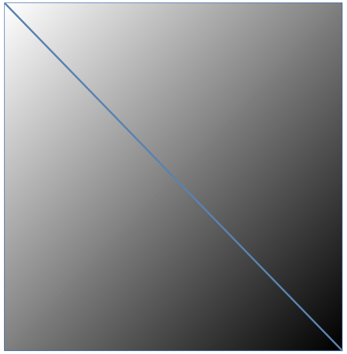




# Connectivity of **G**(*n*,*W*)

- First Obstacle: Disconnectedness of *W*
- Second Obstacle: Isolated vertices

$W(x,y)=x^6y^6$



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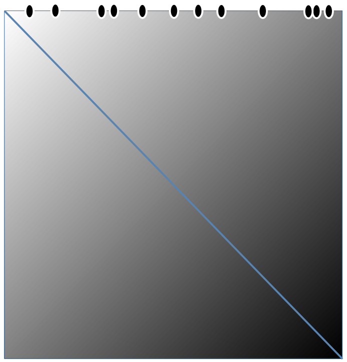

The left-most point $\min\{x_i\} \sim 1 / n < 1/\sqrt{n}$

Hence, $W(x_i,y)<n^{-3}$ for every $y$.

There are $n$-1 people and each of them becomes your friend with probability $<n^{-3}$. 😞

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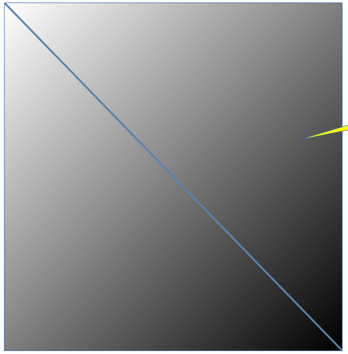

For $x\in[0,1]$, $\deg(x)=\int W(x,y)\,dy$

The left-most …

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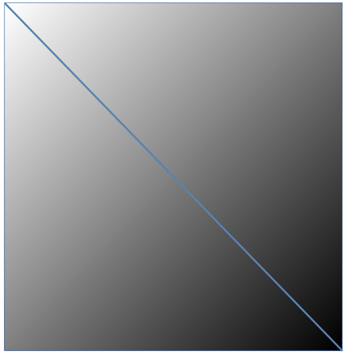
# Connectivity of $\mathbf{G}(n,W)$

- First Obstacle: Disconnectedness of $W$
- Second Obstacle: Isolated vertices

$W(x,y)=x^6y^6$

What about $W(x,y)=x^fy^f$ for $f\in(0,\infty)$?
transition at $f=1$



The left-most point $\min\{x_i\} \approx 1/n < 1/\sqrt{n}$

Hence, $W(x_i,y)<n^{-3}$ for every $y$.

There are $n$-1 people and each of them becomes your friend with probability $<n^{-3}$. 😞

# Connectivity of **$G(n,W)$**

## Theorem [H.-Viswanathan]

Suppose that $W$ is a graphon.

- If $W$ is disconnected (as a graphon) then $\mathbf{G}(n,W)$ is disconnected (as a graph) a.a.s.
- If $W$ is connected then for $\alpha \in [0,1]$, write $s(\alpha) \in [0,1]$ for the measure of elements $x$ with $\deg(x) < \alpha$.
  - If $\lim_{\alpha \to 0} s(\alpha) / \alpha = 0$ then a.a.s. $\mathbf{G}(n,W)$ is connected.
  - If $\lim_{\alpha \to 0} s(\alpha) / \alpha = \infty$ then a.a.s. $\mathbf{G}(n,W)$ has an isolated vertex.
  - If $\lim_{\alpha \to 0} s(\alpha) / \alpha \in (0,\infty)$ then connected AND disconnected with prob>0.

# Clique number of $\mathbf{G}(n,W)$

- Theorem [Grimmett-McDiarmid 1975, Matula 1976]: Let $\varepsilon>0$ and $p\in(0,1)$ be fixed. The clique number a.a.s. satisfies

$$\omega(\mathbf{G}(n,p))=(2 \pm \varepsilon)\ln n / \ln(1/p).$$

- Hence: Suppose that $W$ is a graphon so that $0.01 \leq W(x,y) \leq 0.99$. Then

$$\mathbf{G}(n,0.01) \subseteq \mathbf{G}(n,W) \subseteq \mathbf{G}(n,0.99) \quad \text{(stochastic domination)}$$

and so, a.a.s.

$$(2 \pm \varepsilon)\ln n / \ln(1/0.01) \leq \omega(\mathbf{G}(n,W)) \leq (2 \pm \varepsilon)\ln n / \ln(1/0.99)$$

- Goal: find $C_W\in(0,\infty)$ such that $\omega(\mathbf{G}(n,W))=(C_W \pm \varepsilon) \ln n$

# Clique number of **G**($n$,$W$)

- Calculations in the Erdős–Rényi case **G**($n$,$p$). (only 1st moment)

(a) $k = c \ln n,\ c \in (0,\infty)$ to be chosen later

(b) Random variable $X$ counts cliques of size $k$

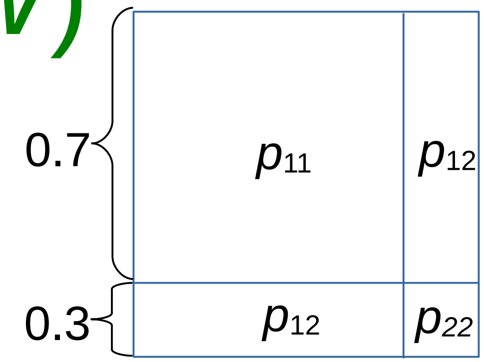
(c)

$$\mathbf{E}X = \binom{n}{k} p^{\binom{k}{2}} \approx n^{c\ln(n)} p^{(c^2 \ln(n)^2/2)} = \exp\left(c\ln(n)^2 + \ln(p)c^2\ln(n)^2/2\right) \begin{cases} \to 0 \\ \to \infty \end{cases}$$

?

$$1 + \ln(p)c/2 \begin{cases} <0 \\ >0 \end{cases}$$

# Clique number of $\mathbf{G}(n,W)$




- A graphon with two steps.

(a) $k = (c_1+c_2)\ln n$, later maximize $c_1+c_2$

(b) Random variable $X$ counts cliques with $c_i\ln n$ vertices in $i$-th block

(c)

$$\mathbf{E}X = \binom{0.7n}{c_1 \ln n}(p_{11})^{\binom{c_1\ln(n)}{2}} \times \binom{0.3n}{c_2\ln n}(p_{22})^{\binom{c_2\ln(n)}{2}} \times (p_{12})^{(c_1\ln(n)c_2\ln(n))} \begin{cases}\to 0\\ \to\infty\end{cases}$$

$$c_1+c_2+\frac{1}{2}\left((c_1)^2\ln(p_{11})+(c_2)^2\ln(p_{22})+2c_1c_2\ln(p_{12})\right) \begin{cases}<0\\ >0\end{cases}$$

# Clique number of $\mathbf{G}(n,W)$

$k = (c_1+c_2)\ln n,$ maximize $c_1+c_2$

$$\underbrace{c_1+c_2}_{\text{entropy}}+\underbrace{\frac{1}{2}\left((c_1)^2\ln(p_{11})+(c_2)^2\ln(p_{22})+2c_1c_2\ln(p_{12})\right)}_{\text{energy}} > 0$$

# Clique number of $\mathbf{G}(n,W)$

## Theorem [Doležal, H., Máthé]

Suppose that $W$ is a graphon. We have

$$\omega(\mathbf{G}(n,W))=(C_W \pm \varepsilon)\ln n.$$

where

$$C_W = \sup_f \int f(x)\,dx$$

and the supremum ranges over all functions $f$:[0,1]→[0,∞) such that

$$\underbrace{\int_x f(x)\,dx}_{\text{entropy}} + \underbrace{\frac{1}{2}\int_x\int_y f(x)f(y)\ln(W(x,y))\,dy\,dx}_{\text{energy}} > 0$$