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FOREWORD

The Fourteenth Annual Summer Research Institute, sponsored by the American Mathematical Society and the Association for Symbolic Logic, was devoted to Axiomatic Set Theory. Financial support was provided by a grant from the National Science Foundation. The institute was held at the University of California, Los Angeles, from July 10 to August 5, 1967, and was attended by more than 125 participants. The Organizing Committee consisted of Paul J. Cohen, Abraham Robinson (chairman), and Dana S. Scott (editor). Special thanks are due to the Department of Mathematics of UCLA for providing facilities and assistance which contributed in large measure to the excellent success of the meeting.

The program for the four weeks of the institute was organized into two ten-lecture series, given by Dana S. Scott and Joseph R. Shoenfield, plus individual contributions generally in one-hour sessions at the rate of four lectures per day. By the last week this was reduced to three per day, as the strength of the participants had noticeably weakened. Nevertheless, most of the success of the institute was due to the fact that nearly everyone attended all of the sessions.

The papers in this volume of the proceedings represent revised and generally more detailed versions of the lectures presented at the institute. In view of the large number of papers, which resulted in delaying the receipt of papers from some authors, it was felt advisable to divide the proceedings into two volumes so as not to delay the publication of these papers any longer.

DANA S. SCOTT

SETS, SEMISETS, MODELS

PETR HÁJEK

This is an expository article written for two purposes. First, it is to give a survey of works of the members of the Prague seminar on foundations of set theory and a full bibliography of them. Secondly, it is to explicate the matter of interest from the present point of view and, in fact, to summarize the contents of a monograph being written by P. Vopěnka and the author of the present paper. These two purposes are followed simultaneously throughout the paper; the author would be glad if the paper was of some help as a guide for reading papers mentioned in the bibliography.

Speaking on the study of foundations of the set theory in Czechoslovakia we must begin with the name of the late Professor L. Rieger. He was the first Czechoslovak mathematician to work on this field. (See Czech. Math. J. (89) **14** (1964), 629 ff. for a short account of his life and papers.) After his tragic death in 1963, his student and fellow-worker P. Vopěnka founded a seminar and engaged the attention of several young people for the study of foundations. Now, after five years, the seminar consists of the following members: B. Balcar, L. Bukovský, K. Hrbáček, T. Jech, A. Sochor, P. Štěpánek, P. Vopěnka and the author.

Let us begin with a trivial remark. Studying metamathematics, it is not uniquely determined which intuitive concepts are presupposed to be sufficiently known. In the case of the syntax of axiomatic theories, in our case of the set theory (or of theories of sets), the notion of a finite sequence of symbols and that of an effective (decidable) system of these sequences may suffice. There are at least two reasons for such a minimization of means: the metamathematical one, consisting in the fact that the finitary conception of the syntax gives to our metamathematical study more "anthropological" character and enables us to answer adequately the question of what the mathematicians can do (prove, decide) and what they cannot. Secondly, there is a mathematical reason, consisting in the fact that, from the mathematical point of view (i.e. from the point of view of developing a particular axiomatic theory) finitary metamathematical results may be consequently understood as auxiliary principles for obtaining new proofs or notions in the theory we are dealing with. Thus, we keep this finitary intuitive conception, being interested in foundations both as logicians and as mathematicians. E.g. a statement "there is a formula . . ." is demonstrated if and only if an effective method is given for finding such a formula. Speaking on a mapping in the metalanguage we always assume a method to be given which enables to find the image of every element to be mapped.

Even if not formulated explicitly, that which is said above has been our point of view from the beginning. But, because of the simultaneous metamathematico-mathematical interest, parts written in the object language and those written in the metalanguage are sometimes not distinguished clearly in earlier papers from the bibliography. The reader is suggested to read those works on the basis of the present paper.

As we are interested in the study of concrete theories (namely, the Gödel-Bernays set theory and some related theories) we are forced to choose the basic formal system quite rich and thus quite near to actual mathematical techniques. But it need not be explained in details. Imagine simply, we have *variables* and *constants* of various *sorts*, *predicates* of various *ranks*, and *operations* of various *sorts* and *ranks*. One sort is preferred as *universal*. *Terms* and *formulas* are defined in the usual way; any (finite) sequence of formulas may be considered as an *axiomatics*. The *language* of an axiomatics is the (finite) list of all predicates, constants, operations and sorts of variables occurring in the axioms. (A sort may be represented by an arbitrary variable of this sort.) The notion of *logical axioms* (tautologies) and *deductive rules* is defined; thus, we have a formal notion of a *proof* in an axiomatic theory. (*Theory* is given by its axiomatics.) If $\mathcal{T}$ is a theory, then a formula is a *$\mathcal{T}$-formula* iff it is formulated by means of things occurring in the language of $\mathcal{T}$; a $\mathcal{T}$-formula is *$\mathcal{T}$-provable* (denotation: $\mathcal{T} \vdash \varphi$) iff there is a proof of it from the axioms of $\mathcal{T}$. A theory $\mathcal{S}$ is an *extension of $\mathcal{T}$* iff the sequence of axioms of $\mathcal{T}$ is a segment of the sequence of axioms of $\mathcal{S}$. The notion of a *contradictory* and *consistent* theory (introduced in this order) is usual.

A mapping $\mathcal{M}$ of $\mathcal{T}$-formulas into $\mathcal{S}$-formulas is a *syntactic model* of $\mathcal{T}$ in $\mathcal{S}$ iff

(a) $\mathcal{M}$ respects both logical axioms and the axioms of $\mathcal{T}$, i.e. maps these axioms into $\mathcal{S}$-provable formulas,

(b) $\mathcal{M}$ respects deduction rules, i.e. if a $\mathcal{T}$-formula immediately follows from some $\mathcal{T}$-formula(s) (assumption(s)) then the image of the former formula is provable in the extension of $\mathcal{S}$ by the image(s) of the assumption(s);

(c) $\mathcal{M}$ respects the negation, i.e. the negation of the image of a $\mathcal{T}$-formula is provable in the extension of $\mathcal{S}$ by the image of the negation of that formula.

Provability principle. Let $\mathcal{M}$ be a model of $\mathcal{T}$ in $\mathcal{S}$, then the image of every $\mathcal{T}$-provable formula is $\mathcal{S}$-provable.

Consistency principle. If $\mathcal{T}$ has a model in $\mathcal{S}$ and $\mathcal{T}$ is contradictory then $\mathcal{S}$ is too; a fortiori, if $\mathcal{T}$ has a model in $\mathcal{S}$ and $\mathcal{S}$ is consistent then $\mathcal{T}$ is too.

A $\mathcal{T}$-formula φ is said to *hold* in a model $\mathcal{M}$ of $\mathcal{T}$ in $\mathcal{S}$ iff $\mathcal{S} \vdash \varphi^{\mathcal{M}}$ ($\varphi^{\mathcal{M}}$ being the image of φ by $\mathcal{M}$). A model $\mathcal{M}_1$ is *stronger* than $\mathcal{M}_2$ iff every formula holding in $\mathcal{M}_2$ holds in $\mathcal{M}_1$ too. $\mathcal{M}_1$ is *equivalent* to $\mathcal{M}_2$ iff $\mathcal{M}_1$ is stronger than $\mathcal{M}_2$ and $\mathcal{M}_2$ is stronger than $\mathcal{M}_1$. Identical mapping of $\mathcal{T}$-formulas is called the *identical model* of $\mathcal{T}$. If $\mathcal{M}_1$ is a model of $\mathcal{T}_1$ in $\mathcal{T}_2$ and $\mathcal{M}_2$ is a model of $\mathcal{T}_2$ in $\mathcal{T}_3$, then the composed mapping is denoted by $\mathcal{M}_1 * \mathcal{M}_2$ and called *$\mathcal{M}_1$ constructed in $\mathcal{M}_2$*. Theories $\mathcal{T}$ and $\mathcal{S}$ are *equivalent* iff there are models $\mathcal{M}_1$ of $\mathcal{T}$ in $\mathcal{S}$ and $\mathcal{M}_2$ of $\mathcal{S}$ in $\mathcal{T}$ such that $\mathcal{M}_1 * \mathcal{M}_2$ is equivalent to the identical model of $\mathcal{T}$ and $\mathcal{M}_2 * \mathcal{M}_1$ is equivalent to the identical model of $\mathcal{S}$.[1]

EXAMPLES. (1) There may be defined explicitly what a definition of a predicate, constant, operation, sort of variables respectively in a theory is. An extension of a theory by adding such a definition is equivalent to the original theory. (2) More generally for constants: Let $\mathcal{T} \vdash (\exists x_1, \ldots, x_n)\pi(x_1, \ldots, x_n)$. If we add the axiom $\pi(a_1, \ldots, a_n)$ where a's are new constants we obtain an equivalent theory. (In this case we say that we have fixed the parameters with the help of π.)

Now it is possible to formulate the axioms of the fundamental gödelian theory of classes **TC** and describe a very general kind of models of **TC** in itself. We are interested in developing this theory from a unary predicate $\in$ and variables of the universal sort only (the language $(\in, X)$ is called the *fundamental language*); all other notions, including the equality predicate, are defined. (This possibility has been observed and used by several authors.) Instead of doing it explicitly we only give the definitions of the notions we need; the axioms serve only to the fact that the following definitions really are definitions in the sense of the calculus.

$X = Y \equiv (\forall Z)(Z \in X \equiv Z \in Y)$	equality predicate
$(\exists x)(x = X) \equiv (\exists Z)(X \in Z)$	set variables
$z \in \{X, Y\} \equiv .z = X \vee z = Y$	pairing operation
$\langle x, y\rangle = \{\{x\}, \{x, y\}\}$	ordered pair operation
$(\forall x)(x \in V)$	constant for universal class
$x \in \mathfrak{E}(X) \equiv (\exists u, v)(x = \langle u, v\rangle \,\&\, x \in X \,\&\, u \in v)$	$\in$-representation on X (operation)
$x \in X - Y \equiv .x \in X \,\&\, x \notin Y$	difference
$x \in \mathfrak{D}(X) \equiv (\exists y)(\langle y, x\rangle \in X)$	domain
$x \in X \upharpoonright Y \equiv .x \in X \,\&\, (\exists u, v)(x = \langle u, v\rangle \,\&\, v \in Y)$	restriction
$x \in \mathfrak{Cn}(X) \equiv (\exists u, v)(x = \langle u, v\rangle \,\&\, \langle v, u\rangle \in X)$	conversion
$x \in \mathfrak{Cn}_3(X) \equiv (\exists u, v, w)(x = \langle u, v, w\rangle \,\&\, \langle v, w, u\rangle \in X)$	ternary conversion

[1] By the way, if we gave up our finitary point of view, it could be of some interest to deal with the category of theories as objects and (some) syntactic models as morphisms.

The operations $\{\ \}$, $\mathfrak{E}$, $-$, $\mathfrak{D}$, $\restriction$, $\mathfrak{Cn}$, $\mathfrak{Cn}_3$ are called *gödelian operations* and sometimes denoted by $\mathfrak{F}_1(X, Y), \ldots, \mathfrak{F}_7(X, Y)$ (in this order). A term built up from the constant V, universal variables and the operations $\mathfrak{F}_2, \ldots, \mathfrak{F}_7$ is called a *gödelian term*. A formula built up from the predicate $\in$, universal variables (called also class variables), and set variables in which no class variable is bound is called *normal*. A well-known metatheorem on normal formulas may be stated in the following way:

Let $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$ be normal. Then there is a gödelian term $\mathfrak{T}(X_1, \ldots, X_m)$ such that

$$\mathbf{TC} \vdash \langle x_1, \ldots, x_n\rangle \in \mathfrak{T}(X_1, \ldots, X_n) \equiv \varphi(x_1, \ldots, x_n, X_1, \ldots, X_m).$$

Even if we have not written down the axioms of **TC** we shall use some names of them. Besides two auxiliary axioms F1, F2 we have an axiom A1 "justifying the definition of the ordered pair", B1 "justifying the definition of the universal class" and B2 – B7 "justifying the definitions of the operations $\mathfrak{F}_2 - \mathfrak{F}_7$". (E.g. B1 is $(\exists Z)(\forall x)(x \in Z)$ etc.)

A *theory of classes* is any extension of **TC** such that every new axiom either is formulated in the language of the preceding segment (is a proper axiom) or is a definition of a new concept.[2]

Fundamental formulas are formulas of the fundamental language, i.e. those built up from the predicate $\in$ and class variables. Set formulas are formulas built up from $\in$ and set variables.

Fundamentalization principle. Let $\mathcal{T}$ be a theory of classes. Then there is a mapping $\mathcal{F}$ associating with every $\mathcal{T}$-formula φ a fundamental formula $\varphi^{\mathcal{F}}$ deductively equivalent to φ in $\mathcal{T}$; moreover, the sequence of images of proper axioms of $\mathcal{T}$ is a theory $\mathcal{T}^{\mathcal{F}}$ equivalent to $\mathcal{T}$ and $\mathcal{F}$ is a model of $\mathcal{T}$ in $\mathcal{T}^{\mathcal{F}}$. $\varphi^{\mathcal{F}}$ is called the fundamentalization of φ.

Given a theory of classes $\mathcal{T}$ and another theory $\mathcal{S}$, given further a binary predicate $\in^*$ and a sort of variables $X^*, Y^*, \ldots$ in the language of $\mathcal{S}$, we may define a mapping of $\mathcal{T}$-formulas into $\mathcal{S}$-formulas as follows; for every $\mathcal{T}$-formula φ, take its fundamentalization $\varphi^{\mathcal{F}}$ and, in the latter formula, put $\in^*$ instead of $\in$, X^* instead of X, etc. This mapping is denoted by $\mathcal{I}m(\mathcal{T}, \mathcal{L})$, where $\mathcal{L}$ is the language $(\in^*, X^*)$ (read: the *imitation* of the $\mathcal{T}$-formulas given by $\mathcal{L}$); this mapping is a model of $\mathcal{T}$ in $\mathcal{S}$ iff images of proper axioms of $\mathcal{T}$ are provable in $\mathcal{S}$. (This notion is closely related to the Tarski's notion of relative interpretability; cf. also [18], [42].)

Sometimes we do not have such a language $(\in^*, X^*)$ (which may be called an F-like language) but we are able to introduce it. In the optimal case, we find two formulas $\chi(X)$ and $\varepsilon(X, Y)$ such that we are able to define X^* as those X that $\chi(X)$ and to define $\in^*$ with help of ε. The couple of formulas χ, ε is called a *nonparametric basis* (of an F-like language). A triple $\mathcal{B}$ of formulas $\pi(\boldsymbol{u})$, $\chi(X, \boldsymbol{u})$, $\varepsilon(X, Y, \boldsymbol{u})$

[2] Notions which are used in the usual sense will not be defined here, as the power class, the field of a relation etc.

($\boldsymbol{u}$ being a finite sequence of variables) is called a *parametric basis* iff $\mathcal{S} \vdash (\exists \boldsymbol{u})\pi(\boldsymbol{u})$ and $\mathcal{S} \vdash \pi(\boldsymbol{u}) \rightarrow (\exists X)\chi(X, \boldsymbol{u})$. Having such a basis we may first fix parameters with help of π ("take arbitrary but fixed $\boldsymbol{a}$ such that $\pi(\boldsymbol{a})$") and then define X^* as those X that $\chi(X, \boldsymbol{a})$ and $X^* \in^* Y^*$ as $\varepsilon(X, Y, \boldsymbol{a})$. (We obtain a theory $\mathcal{S}_1$ equivalent to $\mathcal{S}$ in this way.) Let us write $\mathcal{I}m(\mathcal{T}, \mathcal{B})$ instead of $\mathcal{I}m(\mathcal{T}, \mathcal{L})$ where $\mathcal{L}$ is $(\in^*, X^*)$. Or we may associate with every $\mathcal{T}$-formula not containing the variables $\boldsymbol{u}$ (more precisely: with its fundamentalization) a formula $\varphi^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})}$ by the following induction: $(X \in Y)^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})}$ is $\varepsilon(X, Y, \boldsymbol{u})$, $(\varphi \ \& \ \psi)^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})}$ is $\varphi^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})} \ \& \ \psi^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})}$ etc., $[(\forall X)\varphi(X)]^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})}$ is $(\forall X)(\chi(X, \boldsymbol{u}) \rightarrow [\varphi(X)]^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})})$ etc. (The mapping $\mathcal{P}ar$ is called the *parametric imitation* of $\mathcal{T}$-formulas given by $\mathcal{B}$.) If φ is closed then $\pi(\boldsymbol{a}) \rightarrow \varphi^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})}(\boldsymbol{a})$ is deductively equivalent to $\varphi^{\mathcal{I}m(\mathcal{T}, \mathcal{B})}$ in $\mathcal{S}_1$ (one can say even more); but in any case, $\varphi^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})}$ has at least free variables $\boldsymbol{u}$; to every closed φ a notion concerning $\boldsymbol{u}$ is associated in this way. Given a concrete basis, we write usually φ^* instead of $\varphi^{\mathcal{I}m(\mathcal{T}, \mathcal{B})}$ and $\varphi^{\boldsymbol{u}}$ instead of $\varphi^{\mathcal{P}ar(\mathcal{T}, \mathcal{B})}$. ($\varphi^{\boldsymbol{u}}$ is often read "φ holds in sense of $\boldsymbol{u}$".) We also write $\mathrm{Cls}^{\boldsymbol{u}}(X)$ instead of $\chi(X, \boldsymbol{u})$.

Let $\mathcal{B} = (\pi(\boldsymbol{u}), \chi(X, \boldsymbol{u}), \varepsilon(X, Y, \boldsymbol{u}))$ be a basis in $\mathcal{S}$. A basis $\mathcal{B}' = (\pi')(\boldsymbol{u})$, $\chi(X, \boldsymbol{u})$, $\varepsilon(X, Y, \boldsymbol{u})$ is called a *specification* of $\mathcal{B}$ (in $\mathcal{S}$) iff $\mathcal{S} \vdash \pi'(\boldsymbol{u}) \rightarrow \pi(\boldsymbol{u})$. If $\mathcal{B}$ determines a model of $\mathcal{T}$ and $\mathcal{B}'$ is a specification of $\mathcal{B}$ then $\mathcal{B}'$ determines a model of $\mathcal{T}$ stronger than the former one.

Very important example for **TC**. Define in **TC**: *Relation* is a class of ordered pairs. *Extension* of x in a relation R (denoted $\mathfrak{Ext}_R(x)$) is the class $\{y; \langle y, x\rangle \in R\}$. R is an *E-like relation* (denotation: $\mathrm{Elk}(R)$) iff R is a nonempty relation, is *internal* (i.e. for any $x, y \in \mathit{field}(R)$, $\mathfrak{Ext}_R(x) = \mathfrak{Ext}_R(y)$ implies $x = y$) and *closed* with respect to the 1st operation (i.e. for every $x, y \in \mathit{field}(R)$, there is a z such that $\mathfrak{Ext}_R(z) = \{x, y\}$).

The triple $\mathcal{N}_{\mathrm{TC}}(R)$ of formulas

$$\mathrm{Elk}(R), \qquad X \subseteq \mathit{field}(R), \qquad (\exists z \in Y)(X = \mathfrak{Ext}_R(z))$$

is a basis which determines a model of **TC** in **TC**. E.g. the relation $E = \mathfrak{E}(V)$ is an E-like relation.

Classes of the model are all subclasses of the field of R; elements of the field of R are codes of sets of the model, the membership is determined by R (X belongs to Y in sense of the model iff the code of X belongs to Y). Now, we are interested in theories stronger than **TC** and their models. Evidently, the basis $\mathcal{N}_{\mathrm{TC}}(R)$ determines a model of **TC** in every theory stronger than **TC**; but if we specify the conditions on R, we obtain more. We are just going to discuss this.

We shall deal with two extensions of **TC**: with the theory of sets **TS** and the theory of semisets **TSS**. The former one is equivalent to Gödel's theory with axiom groups **A**, **B**, **C** and corresponds to **ZF** without the axiom of regularity. The latter one is weaker than **TS** and was formulated by Vopěnka in Summer 1967 in cooperation with the author; this theory is explained here with the permission of P. Vopěnka. *Semisets* are defined in **TC** as subclasses of sets. ($\mathrm{Sm}(X) \equiv (\exists y)(X \subseteq y)$.) This notion is superfluous in **TS**, as all semisets are sets; but it is of a fundamental importance in **TSS**. As we shall see, there are two ways of constructing models of **TS** in **TSS** (with some additional axioms): going below, i.e. omitting some proper classes (nonsets), and going above, i.e. make semisets to sets. It enables us e.g. to demonstrate various forms of the independence of the axiom of constructibility by constructing models of **TSS** in **TS** (which is not difficult) and then working in **TSS**. From this point of view, **TSS** is the key means of the latest Vopěnka's reformulation of Cohen's method; the advantage consists in the fact that main proofs are done not about model-sets etc., say, in the model, but simply in **TSS**. See below for more details.

Relations will play a prominent rôle in the sequel. Let us define some notions about them (in **TC**). R is called nowhere constant iff, for any $x, y \in \mathfrak{D}(R)$, $\mathfrak{Ext}_R(X) = \mathfrak{Ext}_R(Y)$ implies $X = Y$. If R is a relation, then R is called *regular* iff all extensions are semisets, i.e. $(\forall x \in \mathfrak{D}(R))(\exists y)(\mathfrak{Ext}_R(x) \subseteq y)$. Regular nowhere constant relations may be considered as 1-1 associations of semisets to sets; it helps to formulate some axioms. Axioms of **TSS**: those of **TC** plus

(A2)–(A7) $M(\mathfrak{F}_i(x, y))$ $i = 2, \ldots, 7$ (gödelian operations make sets from sets)

(C1) $(\exists x \neq 0)(\forall y \in x)(\exists z \in x)(y \subset z)$ (infinity)

(C2) R regular nowhere constant $\rightarrow (\mathrm{Sm}(\mathfrak{D}(R)) \equiv \mathrm{Sm}(\mathfrak{W}(R)))$

($\mathfrak{W}(R)$ is the domain of values, i.e. $\{y; (\exists x)(\langle yx\rangle \in R)\}$). Axioms of **TS**: those of **TC** plus (C1), (C2), plus

(C3) every semiset is a set.

The axioms (A2)–(A7) are provable in **TS**, hence **TS** is stronger than **TSS**. Further, the original Gödel's axioms (C2)–(C4) are provable in **TS** and our axioms (C2)–(C3) are provable in Gödel's axiomatics. But we cannot replace (C2) as an axiom of **TSS** by Gödel's (C2), (C3) (power set and sum set axioms). The power class and sum class of a set is proved to be a set in **TSS**. In **TS**, (C2) may be evidently replaced by

(C2′) R regular nowhere constant $\rightarrow (M(\mathfrak{D}(R)) \equiv M(\mathfrak{W}(R)))$

(M being the predicate "... is a set").
First axiom of regularity:

(D1) $(\forall X)(\forall x)(\exists y)(\mathfrak{D}(X) \cap x = \mathfrak{D}(X \cap y))$.

Equivalently: For every relation whose domain is a semiset there is a subrelation which is a semiset and has the same domain.

In **TSS** + (D1), the comprehension schema is provable. A class X is said to have *set intersection property* (SIP(X)) iff, for every set x, $X \cap x$ is a set. Define new variables X^* for classes with set intersection property, define $X^* \in^* Y^* \equiv X^* \in Y^*$. In this way, we obtain a model of **TS** + (D1) in **TSS** + (D1) with absolute notion of set. As a consequence, we obtain the

Equiprovability principle. Let φ be a set formula. Then **TSS** + (D1) $\vdash \varphi$ if and only if **TS** + (D1) $\vdash \varphi$.

The model just described is called the natural model and denoted by $\mathcal{N}at$.

Consider the basis $\mathcal{N}_{\mathrm{TC}}(R)$ defined above. We write φ^R instead of $\varphi^{\mathcal{V}ar(\mathbf{TSS}, \mathcal{N}_{\mathrm{TC}}(R))}$. Define (in **TSS**): Let R be an E-like relation. (a) R is *almost universal* iff every semiset included in the field of R is included in the extension of a $x \in \mathfrak{D}(R)$; (b) R is *closed* with respect to the ith operation iff $(\mathrm{A}i)^R$ holds ($i = 2, \ldots, 7$; i e. the axiom (Ai) holds in the sense of R); R is *closed* iff it is closed with respect to all 7 operations. (c) R is *relatively infinite* iff $(\mathrm{C1})^R$ holds. (d) R is a *model relation* (Mrel(R)) iff it is regular, internal, closed, relatively infinite and almost universal.

METATHEOREM. *The basis*

$$\mathrm{Mrel}(R), \qquad X \subseteq \mathit{field}(R), \qquad (\exists z \in Y)(X = \mathfrak{Ext}_R(z))$$

(it is a specification of $\mathcal{N}_{\mathrm{TC}}$*) determines a model of* **TSS** *in* **TSS** *and of* **TSS** + (D1) *in* **TSS** + (D1). *(Cf.* **[2]**, **[8]**).

In other words, we can prove in **TSS** (+ (D1)) that all axioms of **TSS** (+ (D1)) hold in sense of every model relation. A fortiori, we can prove the same in **TS** (+ (D1)) but we cannot prove (C3)) in sense of every model relation in **TS**. The basis just described is called the *normal basis* for **TSS** and denoted by $\mathcal{N}_{\mathrm{TSS}}(R)$; also the corresponding model is denoted by $\mathcal{N}_{\mathrm{TSS}}(R)$. The composed model $\mathcal{N}at * \mathcal{N}_{\mathrm{TSS}}(R)$ is called the *normal model* of **TS** + (D1) in **TSS** + (D1). The theory **TSS** + (D1) will be denoted by **TSS′**.

Model relations of the form $E \cap P$, E being the original membership relation, and P being a transitive class, are of particular interest. We define (in **TSS**): P is a model class (Mcl(P)) iff it is transitive, contains with arbitrary x, y results of all gödelian operations from them and, for every semiset $X \subseteq P$, there is an $x \in P$ such that $X \subseteq x$.

THEOREM (**TSS**). *If* P *is a model class then* $E \cap P$ *is a model relation.*[3]

Hence if we want to construct models of **TS** in **TS** + (D1) it suffices to prove the existence of various model relations in the latter theory. Cf. **[34]**.

On the other hand, a mathematical theory of model-relations and model-classes may be developed within the theory of (semi)sets; e.g. we may study the structure of model classes as interesting set theoretical objects not speaking about metamathematical problems.

Let us also give a slight generalization of the notion of a model relation. Define (in **TSS**): R is a *weak model relation* (wMrel(R)) iff there is a model relation R_0 and a function F such that F maps the field of R onto the field of R_0 and $(\forall x, y \in \mathit{field}(R))(\langle x, y\rangle \in R \equiv \langle F(x), F(y)\rangle \in R_0)$. An $X \subseteq \mathit{field}(R)$ is called *saturated* iff, for every $x \in X$, $y \in \mathit{field}(R)$, $\mathfrak{Ext}_R(x) = \mathfrak{Ext}_R(y)$ implies $y \in X$.

[3] This theorem may be used also for **ZF** as a metatheorem: let a class P be defined, let **ZF** $\vdash$ (1) $x, y \in P \rightarrow \mathfrak{F}_i(x, y) \in P$, (2) $x \subseteq P \rightarrow (\exists y \in P)(x \subseteq y)$, (3) P is transitive. Then P is a model for **ZF**.

(Denotation: $\mathrm{Sat}_R(X)$.) The basis

$$\mathrm{wMrel}(R), \qquad X \subseteq \mathit{field}(R) \;\&\; \mathrm{Sat}_R(X), \qquad (\exists z \in Y)(X = \mathfrak{Ext}_R(z))$$

determines a model of **TSS** (+ (D1)) in **TSS** (+ (D1)) equivalent to $\mathcal{N}_{\mathbf{TSS}}(R)$.

Ultraproduct-relations (see below) give examples of weak model relations.

Now, let us deal briefly with **TS**. (We shall come back to **TSS** later; the equiprovability principle will supply a lot of theorems of **TSS**.) Ordinal and cardinal numbers are defined in **TS** in the usual way. If the first axiom of choice (E1) is assumed (every set can be well-ordered), cardinal arithmetic may be studied (cf. [**25**]).

Further (stronger) axioms of regularity and choice are formulated as analogous to each other.

(D2) There is a regular relation R such that $\mathfrak{D}(R) = On$ (the class of all ordinal numbers) and $\mathfrak{W}(R) = V$.

(E2) There is a function F such that $\mathfrak{D}(F) = On$ and $\mathfrak{W}(F) = V$.

Evidently, $\mathbf{TS} \vdash \mathrm{E2} \to \mathrm{D2}$, $\mathrm{D2} \to \mathrm{D1}$, $\mathrm{E2} \to \mathrm{E1}$. Both axioms can be strengthened by defining a relation (function) and postulating that it fulfills the conditions of (D2), (E2) respectively.

(D3) is equivalent to the usual regularity axiom; define $p_0 = 0$ (empty set), $p_{\alpha+1} =$ the power set of p_α, $p_\lambda = \bigcup_{\alpha<\lambda} p_\alpha$ for λ limit number, $Ker = \bigcup_{\alpha \in On} p_\alpha$. (D3) is the assumption $Ker = V$. Defining $\langle x\alpha \rangle \in R \equiv x \in p_\alpha$ we obtain a relation about which we prove in **TS** + (D3): R is regular, $\mathfrak{D}(R) = On$, $\mathfrak{W}(R) = V$. Hence $\mathbf{TS} \vdash \mathrm{D3} \to \mathrm{D2}$. It is well known that (D3) is consistent as it is provable in **TS** that Ker is a model class and (D3) holds in the sense of that model class.

In **TS** + (D3), the following theorem concerning model classes is provable [**49**]: Let M, N be model classes, let the axiom (E1) hold in sense of M, let $\mathfrak{P}^M(On) = \mathfrak{P}^N(On)$ (i.e. M and N have the same sets of ordinals). Then $M = N$.

The assumption that (E1) holds in sense of M is essential, see the paper by Jech in this volume.

The consistency of (E2) can be proved by means of the so called effective model. The model class of hereditarily effective sets equals (provably in **TS** + (D3)) to the model class of hereditarily ordinal-definable sets defined by Myhill and Scott; so we do not describe it in this paper (see this volume for the paper of Myhill and Scott). The axiom (E3) is the assumption that the class of all (hereditarily) effective sets equals V. It implies (E2) but need not hold in sense of all hereditarily effective sets. The Gödel axiom of constructibility $V = L$ is stronger than (E3) and holds in sense of the model class of all constructible sets, so all these axioms are consistent (which is well known). More generally, it is provable in **TS** that, for every X, there is the smallest Z such that $\mathrm{Mcl}(Z)$ and $\mathrm{Cls}^Z(X)$; it is denoted by L_X.

On the other hand, the nonprovability of (D3) is an immediate consequence of the following theorem provable in **TS** + (D1): (see [**14**]; the proof can be radically simplified): Let R be a regular internal relation, then there is a model relation S such that $R \subseteq S$, $(\forall x, y)(\langle x, y\rangle \in S \;\&\; y \in \mathit{field}(R) \to \langle x, y\rangle \in R)$, $u \subseteq \mathit{field}(S) \to (\exists z \in \mathfrak{D}(S))(u = \mathfrak{Ext}_S(z))$.

E.g. the consistency of the existence of a proper class of urelements (sets such that $x = \{x\}$) follows immediately. Classic Fraenkel-Mostowski permutation model classes may be studied. For some particular results see [**36**], [**40**]. It is also possible to show that not only D1 & ¬E1 is consistent but also ¬D1 & E1.

From now, let **TSS*** be the theory **TSS′** + (D3) + (E2), **TS*** be **TS** + (D3) + (E2). We deal with **TS***. The *ultraproduct* weak model relation is defined in dependence on a complete boolean algebra and two other parameters. (This generalization of the usual ultraproduct relation is done by Vopěnka.) Let b be a complete boolean algebra; $Part(b)$ is the set of all disjointed partitions of b. (An $x \subseteq b$ is a *disjointed partition* of b iff

$$(1)\ \bigvee x = 1_b \qquad (2)\ (\forall u, v \in x)(u \neq v \to u \wedge v = 0_b).)$$

$Part(b)$ is a lattice (with respect to the partial ordering by being finer). A set c is a *partitive structure* on b iff it is a filter on $Part(b)$, or c equals to $Part(b)$. Let, moreover, z be an ultrafilter on b; we define $f \in Ulc(b, c)$ (ultraproduct class) iff there is an $x \in c$ such that f is a mapping with the domain x (values arbitrary sets); for $f, g \in Ulc(b, c)$, put $\langle f, g\rangle \in Ulr(b, c, z)$ iff $\bigvee\{u \wedge v;\ u, v \in b \;\&\; f(u) \in g(v)\} \in z$ (ultraproduct relation).

THEOREM (**TS***). *If b is a complete boolean algebra, z an ultrafilter on it, c a partitive structure on it, then $Ulr(b, c, z)$ is a weak model relation.*

METATHEOREM. *Let $\varphi(x, \ldots, y)$ be a set formula; then the following is provable in* **TS***: *Let b be a complete boolean algebra, z an ultrafilter on it, c a partitive structure on it, let R be $Ulr(b, c, z)$. Then, for every $f, \ldots, g \in Ulc(b, c)$*

$$\varphi^R(\mathfrak{Ext}_R(f), \ldots, \mathfrak{Ext}_R(g)) \equiv V\{u \wedge \cdots \wedge v;\ u, \ldots, v \in b \;\&\; \varphi(f(u), \ldots, g(v))\} \in z.$$

Considering the power-set of a set x as a complete boolean algebra b with the usual set-theoretical operations, let us write $Part(x)$ instead of $Part(b)$ and speak about partitions of x instead of partitions of b etc. as usual. Functions whose domain is a partition of x can evidently be replaced by functions on x constant on every element of the partition.

It is well known that the ultraproduct model relations may be used for the study of large cardinals (in theories in which we assume such cardinals). We do not mention details here; let us only mention the result of Vopěnka-Hrbáček concerning the fact that the existence of a strongly compact cardinal is inconsistent with the axiom "there is a set a such that $V = L_a$" (see [**41**]). The proof uses two different partitive structures on an appropriate set and corresponding ultraproduct model relations.

We present also two theorems proved by Balcar and Vopěnka (not yet published). Define in **TS***: Let s be an infinite set; an ultrafilter z on s is *uniform* iff every element of z has the cardinality of s. $c_{\aleph_\alpha}$ is the partitive structure on s of all partitions p of s such that the cardinality of p is less than $\aleph_\alpha$. c_z is the partitive structure of all partitions p of s such that $z \cap p \neq 0$. $c_z \dot{\vee} c_{\aleph_\alpha}$ is the partitive structure generated by c_z and $c_{\aleph_\alpha}$. z is *fine* iff $c_z \dot{\vee} c_{\aleph_\alpha}$ is a maximal filter on *Part*(s).

THEOREM. *If* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ *then, for card*$(s) = \aleph_\alpha$, *there are* $2^{\aleph_{\alpha+1}}$ *fine ultrafilters on* s.

If $f, g \in Ulc(s, c)$ then f is said to be *almost equal* to g iff there is an $x \in z$ such that $f \upharpoonright x = g \upharpoonright x$.

THEOREM. *Let* $\aleph_\alpha$ *be regular,* z *a fine ultrafilter on* $\aleph_\alpha$. *Then every mapping of* $\aleph_\alpha$ *into itself either almost equals to a constant function or there is an* $x \in z$ *such that* $f \upharpoonright x$ *is strictly increasing.*

Let us come back to the theory of semisets. There are two important axioms we shall deal with:

(S1) For every nonempty semiset X, there is a $y \in X$ such that $X \cap y = 0$ (regularity for semisets).

(S2) There is an internal regular relation R such that $\mathfrak{D}(R) = V$ and, for every semiset X, there is a y such that $\mathfrak{Ext}_R(y) = X$ (all semisets can be coded by all sets).

E.g. let M be a constant for a model-class in **TS***, let X^* be a variable for subclasses of M and $\in^*$ the usual membership; then the language $(\in^*, X^*)$ determines a model of **TSS** + (S1) + (S2) in the former theory.

Define in **TSS**: A model relation R is an *extension of the theory* (Eth(R)) iff there is a mapping H of V into the field of R such that

(1) $$\mathrm{Mcl}^R(H''V) \;\&\; (\forall x, y)(x \in y \equiv \langle H(x), H(y)\rangle \in R),$$

(2) for every semiset $X \subseteq$ *field*(R), there is a $y \in$ *field*(R) such that

$$X = \mathfrak{Ext}_R(y).$$

THEOREM (**TSS**′ + S1 + S2). *There is an extension* R *of the theory; it is uniquely determined in the following sense: if* R, S *are extensions of the theory then* Ker^R *is isomorphic to* Ker^S *with respect to* R, S.

Hence, the specification of the normal model of **TS** in **TSS**′ + (S1) + (S2) by Eth(R) is a model of **TS** in **TSS**′ + (S1) + (S2) such that the identical model of **TSS**′ + (S1) + (S2) is equivalent to a (transitive) submodel of the former model. In this way, we obtain a model of **TS** in **TSS**′ + (S1) + (S2) "going above".

There is an axiom stronger than (S1) + (S2); it is called the axiom of a support. Define in **TSS**: A semiset X is a *support* iff, for every semiset Y, there is a function f which is a set such that $Y = f^{-1}[X]$.

(Supp) There is a support.

DEFINITION (**TSS**). Let b be a complete boolean algebra (set); a semiset Z is a set-multiplicative ultrafilter on b iff (1) $(\forall x, y \in b)(x \in Z \;\&\; y \geq x \,.\!\rightarrow y \in Z)$, (2) $(\forall x \in b)(x \in Z \vee -x \in Z)$, (3) for every set $a \subseteq Z$, $\bigwedge a \in Z$.

Define: A support is a *boolean support* iff it is a set-multiplicative ultrafilter on a boolean algebra b.

(B Supp) There is a boolean support.

THEOREM (**TSS***). (B Supp) $\equiv$. (Supp) & (S1).

THEOREM (**TSS***). (B Supp) $\rightarrow$. (S1) & (S2).

METATHEOREM. *Let* $\varphi(b)$, ω *be set formulas, let* **TSS***, $\omega \vdash (\forall b)(\varphi(b) \rightarrow b$ *is a complete boolean algebra)* & $(\exists b)\varphi(b)$; *then the formula*

$(\exists b)(\exists Z)(\varphi(b)$ *& Z is a set-multiplicative ultrafilter on b & Z is a support)*

is consistent with **TSS*** + ω.

This is proved using ultraproduct model relations.

METATHEOREM. *Let, under the assumptions of the preceding metatheorem,* ψ *be a set formula provable in* **TSS*** + ω + $(\exists b)(\exists Z)(b$ *is c. b. algebra & Z is a set-multiplicative ultrafilter on b and is a support); then* ψ *is provable in* **TSS*** + ω *(a fortiori, in* **TS*** + ω).

E.g. a proof of the following theorem may be obtained in this way: (see **[53]**)

THEOREM (**TS***). *Every locally nonseparable metric space is a union of an increasing sequence of* $\aleph_1$ *nowhere dense sets.*

THEOREM (**TSS***). *Let* Z_1, Z_2 *be set-multiplicative ultrafilters on a complete boolean algebra b,* Z_1, Z_2 *supports. Then there is an automorphism p of b such that* $p''Z_1 = Z_2$.

The theory of supports can be applied very fruitfully to the study of model classes in **TS***, because—as we already have seen—**TSS** with the axioms (S1), (S2) axiomatizes well the power class of an arbitrary model class in sense of which the axiom of choice holds and makes its extension possible. We define in **TS***: Let M be a model class; a set $x \subseteq M$ is a *support over* M iff, for every $y \subseteq M$, there is a function $f \in M$ such that $y = f^{-1}[x]$. The following theorem is obtained immediately:

THEOREM (**TS***). *Let M be a model class, q a support over M. Then there is a* $b \in M$, *which is a complete boolean algebra in sense of M, and an ultrafilter z on b closed under intersections of all systems* $a \subseteq b$, $a \in M$ *(say, M-multiplicative), which is a support over M.*

Thus, for every definition φ of a complete boolean algebra, we may e.g. suppose consistently in **TS*** (φ a set formula): The algebra defined by φ in sense of the universe of constructible sets has a constructibly multiplicative ultrafilter which is a support over L (and, of course is not constructible). We obtain a lot of consistent axioms in this way; many conditions concerning sets of constructible sets (absoluteness of cardinals, cardinalities of power sets) may be derived from the properties of the algebra b on which we have a support. We may also characterize all model classes with the axiom of choice; they are described by all subalgebras of b. Another result of Vopěnka:

THEOREM (**TS***). *Let $V = L_a$, where $a \subseteq On$, then there is a support over the model class HEf of hereditarily effective sets.*

Now, suppose, we have an arbitrary model class M with the axiom of choice, a $b \in M$ which is a complete boolean algebra in sense of M and an M-multiplicative ultrafilter on b which is a support. The universum $V_M^{(b)}$ of boolean valued functions may be defined in sense of M such that $(\forall f \in M)(f \in V_M^{(b)} \equiv f$ is a mapping of a subset $a \in M$ of $V_M^{(b)}$ into $b)$ (cf. Scott, these Proceedings, part II, and Vopěnka [**50**]). The value $w(f)$ of every $f \in V_M^{(b)}$ in z is defined such that

$$w(f) = \{w(g);\ g \in \mathfrak{D}(f)\ \&\ f(g) \in Z\}.$$

THEOREM (**TS***). *Let z be an M-multiplicative ultrafilter on b (b is a complete boolean algebra in sense of the model class M), let z be a support. Then every set is a value of a boolean valued function from $V_M^{(b)}$. This is to say, boolean valued functions of M code all sets.*

A historical remark should be placed here. After P. J. Cohen had proved the independence of the continuum hypothesis and the axiom of choice in 1963, Vopěnka tried to use his ideas to prove in **TS*** the existence of a model relation in sense of which the continuum hypothesis does not hold; speaking metamathematically, to demonstrate the consistency of the negation of CH with **TS*** in the described straightforward finitary way. It was done in the paper [**12**] published in Russian. Although this paper is closely related to those of Cohen, some new ideas were necessary because—as Sheperdson had proved—it was necessary to construct a non-well-founded relation which is not a set (and is a model relation with the negation of CH) without any assumptions concerning countable models. Then the conception was generalized and simplified in a series of eight papers ([**16**], [**19**], [**20**], [**33**], [**21**], [**31**], [**46**], [**48**]) written partly together with the author of the present paper. It was proved that the number of parameters of the so-called ∇-model relations can be limited to two—a complete boolean algebra and an ultrafilter on it. The whole theory was then presented in the paper [**50**]. The conception of this paper is deeply analogous to that of D. Scott presented at the Los Angeles Summer Institute but, of course, discovered by both authors independently. From the new Vopěnka's point of view, using semisets, boolean valued functions play only auxiliary, even though very important rôle. The advantage consists in the fact that we need only to prove the consistency of the existence of a support on a complete boolean algebra with the theory of semisets (which is done easily using ultraproducts) and then we deal with model classes and supports over them in the set theory, hence in the standard way, having the result about the extension of the theory of semisets. Let us also mention the fact that, in Scott's terminology, the statement "$V^{(B)}$ has a support over $V^{(2)}$" holds in every

boolean valued model. (More precisely but less generally: Assume $V = L$. Then, for every complete boolean algebra, the statement "There is a set x of constructible sets such that, for every set y of constructible sets, there is a constructible function f such that y is the counterimage of x by f" has the boolean value 1.)

For consistency proofs of particular statements using the methods just described, see e.g. [**26**], [**47**], [**51**], [**52**].

Given a normal filter on the group of all automorphisms of b (everything in sense of the model class M) the class of *hereditarily symmetric* boolean valued functions is defined (see [**21**], [**31**] and Scott, these Proceedings, part II). The values of hereditarily symmetric functions form a model class. In this way, model classes in which the axiom of choice fails can be obtained, and consistency of various statements contradicting axiom of choice may be demonstrated. (See e.g. [**39**].)

Consistency proofs for some existential formulas using Fraenkel-Mostowski permutation models can be modified using the method just mentioned. A general method comes from Jech and Sochor (see [**37**], [**38**]). Let us formulate their result in a form modified a little.

Define in **TS**: $p_0(z) = z$, $p_{\alpha+1}(z) =$ the power-set of $p_\alpha(z)$, $p_\lambda(z) = \bigcup_{\alpha<\lambda} p_\alpha(z)$ for λ a limit number, $Ker(z) = \bigcup_{\alpha \in On} p_\alpha(z)$. A definition $\varphi(\alpha)$ of a cardinal number is *good* iff

$$\mathbf{TS} \vdash (\exists!\alpha)(\alpha \text{ cardinal number } \& \ \varphi(\alpha))$$

$$\mathbf{TS},\ Cn = Cn^L \vdash (\forall\alpha)(\forall M)(\mathrm{Mcl}(M) \to \varphi(\alpha) \equiv \varphi^M(\alpha))$$

(Cn being the class of all cardinal numbers, L the model class of all constructible sets). Let α be a constant defined by a good definition of a cardinal number, let $\psi(z)$ be a set formula with only one free variable z. $\psi(z)$ is said to be *α-restricted* iff all quantifiers are restricted onto $p_\alpha(z)$. The formula $(\exists z)\psi(z)$ is said to have a permutation model iff, for a constant λ defined by a good definition of a cardinal number, the following is provable in the theory: $\mathbf{TS} + V = Ker(Ur) + card(Ur) = \lambda$ (Ur is the constant for the class of urelements): "There is a normal filter F on the group of all permutations of urelements such that, if we denote the Fraenkel-Mostowski model class determined by F as M_F, $(\exists z)(z \cap Ur = 0\ \&\ \psi^{M_F}z)$ holds.

The result of Jech-Sochor: Let α be defined by a good definition, let $\psi(z)$ be an α-restricted set formula, let $(\exists z)\psi(z)$ have a permutation model. Then $(\exists z)\psi(z)$ is consistent with **TS** + (D3).

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[4] The results of [**43**] were obtained by K. Hrbáček after he had known the paper of Solovay cited there as [**8**]. The priority for the results on "mild extensions" contained in §3 of [**43**] belongs hence fully to Lévy and Solovay. (Note by K. Hrbáček.)

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