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THE ALTERNATIVE SET THEORY

by Antonin Sochor , Prague

The aim of this paper is to give a brief outline of Alternative Set Theory (AST) . This theory makes possible the synthesis of a number of mathematical disciplines using new methods, and these new approaches are natural from the point of view of AST .

Alternative Set Theory was created by P. Vopěnka and he presented its first version in his seminar in 1973. After the investigation of the consistency of that axiomatic system (by the author [2]) the original system was modified (by P. Vopěnka) and is now called AST . P. Vopěnka developed in AST such basic notions as e.g. natural numbers, infinite powers and real numbers, and proved a large number of fundamental statements and proposed the conception of topology. During the last two years the foundations for the development of mathematics in AST have been laid. Besides P. Vopěnka other members of his seminar, in particular J. Mlček, K. Čuda, J. Chudáček and the author of the present paper also participated by their results in the creation of mathematics in AST . At the same time the metamathematical problems of AST were investigated by the author.

This paper includes only some mathematical and metamathematical results concerning AST selected to show the possibilities of the theory and to explain its relation to the usual set theory. The results concerning model theory in AST (obtained by J. Mlček and the author) are not included at all. The first comprehensive text about AST , including most mathematical results about AST , was written by P. Vopěnka (in Czech). A similar text about the metamathematics of AST is also being prepared.

First let us explain some reasons why we started to deal with AST . At the end of the last century, Cantor developed set theory. Although his theory was inconsistent, it influenced the whole of mathematics in a decisive way. Very soon theories (consistent, we hope) based on Cantor's ideas were constructed - now we have e.g. the Zermelo-Fraenkel, Gödel-Bernays, Morse and New Foundations set theories. We shall speak about all these theories as Cantor's set theories. We can ask whether there existed possibilities to build up another theory that could replace Cantor's set theory and, consequently, whether there were other possibilities to develop mathematics in our century. At first let us mention at least the following two reasons why Cantor's theory was so important and so fruitful:

1) Cantor's theory became the world of mathematics. All theories investigated up to Cantor's time can be considered as parts of set theory. More precisely they have models in Cantor's theory. For some theories (e.g. for the theory of real numbers) their creation was finished only after this modelling. We have an interpretation of infinitesimal calculus in Cantor's theory, too, but Leibniz's and Newton's original ideas had to be reformulated before this modelling. This was necessary since the notion "infinitely small" cannot be naturally modelled in Cantor's theory.

2) Cantor's theory is a theory of infinity. In Cantor's set theory we have actual infinities and moreover Cantor's theory made possible a general investigation and classification of the notion of infinity.

A theory which wants to be an alternative to Cantor's set theory must satisfy these two requirements at least. Our AST is a theory of infinity and contrary to Cantor's is as poor as possible - there are only two infinite powers. Another difference between AST and Cantor's theory consists in the fact that Cantor's set theory places infinity "behind" finite sets and AST places it "among" finite sets. Infinity is represented in our theory by indeterminate (by a set formula), vague parts of finite sets (see the definition of the "countable" class An further in the text).

The problem whether AST fulfils the first requirement is much more complicated. To show that AST could be the world of mathematics in Cantor's time we have to interpret all the theories in question in AST . We hope that this is possible, up to now we have modelled real numbers (more generally we have developed topology in AST). Moreover we are trying not only to model all these theories in AST , but are looking for their natural interpretations (this concerns mainly infinitesimal calculus). And this is the main reason why we started to build up AST .

In AST there are means which are not available in Cantor's theory. For example we have "inaccessible" natural numbers and therefore we can model in AST the notion "infinitely small". This enables us to interpret directly Leibniz's and Newton's ideas. Further we are able to investigate in AST the connection between the continuous and the discrete. From one point of view we can consider a space (and therefore a motion) as discrete and from the second point of view the same space appears as continuous.

Now, what is the connection between AST and nonstandard methods? In some aspects, they are similar e.g. models showing consistency of AST with respect to Cantor's set theory are particular non-well-founded models. On the other hand there are the following two differences at least: At first nonstandard methods deal with models in Cantor's theory and AST is a new axiomatic theory (which can hardly be considered as a precise axiomatization of nonstandard methods). The second difference is even more important. We want to use means which are available in AST to obtain new approaches and new formalizations of notions in an immediate and natural way (and without intermediate steps such as Cantor's set theory and nonwell-founded models as in the case of nonstandard methods).

For every set theory, T , the theory T for finite sets (T_{Fin}) denotes the theory T where we replace the axiom of infinity by its negation.

AST is similar to the theory of semisets (see [3]) in the sense that both admit classes which are subclasses of sets and which are not sets. It is possible to say that AST is some strengthening of the theory of semisets for finite sets (without the axiom C2) . But the main difference is again in what we want to do in AST ; from this point of view, the theory of semisets is very near to Cantor's theory.

Now we shall describe the construction of AST . At first it is a theory of sets, because we want to keep the useful procedures and notions of Cantor's set theory. Our theory is similar to Gödel-Bernays or Morse for we have classes and sets. Sets can be thought of as objects of our investigation and classes can be thought of as our view (approach) to these objects or, in other words, classes can be considered as idealizations of some properties. Our theory has only finite sets, but classes can be infinite. This approach corresponds with one's idea of the real world – all sets as sets of people, houses and so on are finite and only our generalizations and idealizations are infinite, as e.g. the class of all natural numbers, the class of all real numbers and so on. On the other hand there are possibilities to treat some sets (formally finite) as infinite. We have precedents in real life for this, too. For example the number of all atoms on our globe is considered as finite, but it is also possible to consider it as inaccessible.

AST is a theory with one sort of variables – class variables – denoted by $X,\ldots$ and two binary predicates – relationship $\in$ and equality $=$. Sets are defined as members of classes and are denoted by $x,\ldots$

§ 1 The axioms of AST

1) Axiom of extensionality for classes i.e.

$$(\forall X)(X \in Y \equiv X \in Z) \equiv Y = Z$$

2) All axioms of Zermelo-Fraenkel set theory for finite sets.

3) Morse's class existence scheme i.e. for every (including non-normal) formula $\varphi(X)$ we have the axiom

$$(\exists X)(\forall x)(x \in X \equiv \varphi(x))$$

Up to now we have formulated only axioms which are either axioms or are provable in Morse's set theory for finite sets. The following axiom is inconsistent with $\text{Morse}_{\text{Fin}}$ and therefore by accepting our fourth axiom we depart from Cantor's set theory. In $\text{Morse}_{\text{Fin}}$ the statement

$$(*) \qquad X \subseteq x \rightarrow M(X)$$

is provable, on the other hand its negation is provable in AST . Each mathematician is accustomed to the statement (*) and the question is if there are reasons to assume its negation. Vopěnka's argument must be repeated here:

Ch. Darwin teaches us that there is a finite sequence with monkey Charlie as the first element, with Mr. Charles Darwin as the last element and such that each element of the sequence is the father of the

following one. Of course the first element is a monkey and the last element is not a monkey since it is a man. Moreover if some element is a monkey then the following one is a monkey, too. If there existed a set of all the monkeys in our sequence, we would have trivially a contradiction with the statement that Mr. Ch. Darwin is not a monkey (every set of natural numbers has a first element). It is natural therefore to assume that the property "to be a monkey" describes only a class (in this case we do not obtain a contradiction because we do not require that every class of natural numbers has a first element).

Our example is not artificial, such situations are very frequent in real life. Moreover the existence of proper classes which are subclasses of sets enables us to assume that every set is finite and simultaneously to have infinite powers.

We are now going to formulate our fourth axiom. Using axioms 1) – 3) we can define the natural numbers as usual , N denoting the class of all natural numbers. We define the class of all absolute natural numbers by

$$An = \{n \in N \,;\, (\forall X)(X \subseteq n \rightarrow M(X))\}$$

(a natural number is absolute if all its subclasses are sets). Let us recall that in $\text{Morse}_{\text{Fin}}$ we have trivially $An = N$. On the other hand in our theory we accept the axiom

4) Axiom of extension

$$\text{Fnc}(F) \wedge D(F) = An \rightarrow (\exists f)(\text{Fnc}(f) \wedge F \subseteq f)$$

(every function defined on An is a subclass of a function which is a set).

$An \neq N$ follows from this axiom and therefore we get the negation of the statement (*) . The axiom of extension is very strong and one can say that it is the most important axiom of our theory. It enables us to grasp the notion of limit very naturally. Moreover natural numbers which are not absolute can be considered as inaccessible i.e. in some sense infinite. The existence of such natural numbers enables us to model the notion of "infinite small".

Our fifth axiom is the axiom of choice:

5) For every equivalence relation there is a selector.

Since all sets are finite we can prove the existence of a selector for every set equivalence relation from the other axioms. Therefore our axiom of choice gives something new only for proper classes.

The last axiom says how many infinite powers we have. Let $X \mathrel{\hat{\lesssim}} Y$ denote that there is a 1-1 mapping (possibly a proper class!) of X into Y and let $X \mathrel{\hat{\approx}} Y$ stand for $X \mathrel{\hat{\lesssim}} Y \wedge Y \mathrel{\hat{\lesssim}} X$. We can prove using the axiom of extension that $\neg\, An \mathrel{\hat{\approx}} N$. The cardinality of An can be considered as the infinity of "real" natural numbers and the cardinality of N can be considered as the infinity of real numbers (continuum; in the axiomatic system 1) – 5) we can code all subclasses of An by some natural numbers). The last axiom of AST postulates that there are no other infinite cardinalities.

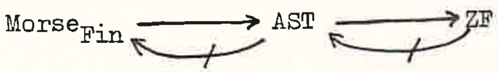
6) Axiom of cardinalities

$$\neg\, X \mathrel{\hat{\lesssim}} An \rightarrow X \mathrel{\hat{\approx}} N$$

We have thence in AST two kinds of proper classes – countable ("small") and the others, the cardinality of which is that of the continuum ("large"). Therefore the axiom of extension can be reformulated in the following form:
Every countable ("small") function is a subclass of a set function. Trivially this statement cannot be true for "large" functions.

§ 2 Metamathematics of AST

We have now described all the axioms of AST. Before we describe what we can do in AST we are going to discuss the consistency of AST and more generally the connection between AST and Cantor's set theory.
We have the following diagram:



where $\longrightarrow$ means that there is an interpretation of the first theory in the second one and $\not\longleftarrow$ means that the interpretation in question does not exist. In this paper we restrict ourselves to sketching a proof of the existence of an interpretation of AST in Zermelo-Fraenkel set theory, a fact which is almost obvious. The following construction is done in Zermelo-Fraenkel.

Let $\underline{M} = \langle M, E \upharpoonright M\rangle$ be the model of all hereditarily finite sets and let Z be a non-trivial ultrafilter on ω_0. Let $\underline{\tilde N} = \langle \tilde N, \tilde E\rangle = \underline{M}^{\omega}/Z$ be the usual ultrapower. $\underline{\tilde N}$ is a model of ZF_{Fin} and to obtain a model containing classes we add "all subclasses of $\underline{\tilde N}$" i.e.

$\underline{\tilde N}' = \langle \tilde N \cup Q, \tilde E \cup (E \upharpoonright Q)\rangle$ where $Q = \{x \subseteq \tilde N ; \neg(\exists f)(x = \{g; \underline{\tilde N} \models g \in f\})\}$

then $\underline{\tilde N}' \models AST$ is provable ; in the following two paragraphs we are going to prove particularly that the axiom of extension and the axiom of cardinalities hold in $\underline{\tilde N}'$.

Let k_x be the constant function the value of which is x. The class of all absolute natural numbers in our model is the set of all constants the values of which are natural numbers i.e. $An^{\underline{\tilde N}'} = \{k_n : n \in \omega_0\}$. To prove the axiom of extension in the model let us suppose $\underline{\tilde N}' \models Fnc(F) \wedge D(F) = An$, then for every $n \in \omega_0$ there is a function f_n such that $\underline{\tilde N}' \models \langle f_n, k_n\rangle \in F$. Let us define a function f on ω_0 by

$$f(n) = \{\langle f_1(n), 1\rangle, \dots, \langle f_n(n), n\rangle\} .$$

We can suppose $f \in \tilde N$ and moreover we have $\underline{\tilde N}' \models Fnc(f) \wedge F \subseteq f$.

To prove the axiom of cardinalities suppose that we had started in $ZF + 2^{\aleph_0} = \aleph_1$. In this case we have $card(An^{\underline{\tilde N}'}) = \aleph_0$ and $card(N^{\underline{\tilde N}'}) = card(\tilde N) = \aleph_1$. Therefore for every infinite $X \subseteq \tilde N$ there is 1-1 mapping between X and either $An^{\underline{\tilde N}'}$ or $N^{\underline{\tilde N}'}$. Hence for $X \in Q$ we have $\underline{\tilde N}' \models X \mathrel{\hat{\approx}} An \vee X \mathrel{\hat{\approx}} N$. If $X = \{g : \underline{\tilde N} \models g \in f\}$ then $\underline{\tilde N}' \models f \mathrel{\hat{\approx}} An \vee f \mathrel{\hat{\approx}} N$ (in fact only the second case can happen). If $X \subseteq \tilde N$ is finite then there are f and $n \in \omega_0$ such that $X = \{g : \underline{\tilde N} \models g \in f\} \wedge \underline{\tilde N}' \models f \mathrel{\hat{\approx}} k_n$.

The following metamathematical results concern independence of the axioms. Our attention is directed mainly to the problems relating to the last three axioms. The results concerning the last axiom are satisfactory: We can prove that the theories obtained, from AST, by the substitution of the last axiom by one of the axioms "there are there (four,... resp.) cardinalities"
"there are cofinally many cardinalities"
are consistent with respect to Zermelo-Fraenkel set theory.

We can also prove that the theory obtained from AST by the substitution of the axiom of extension by its negation and by adding the axiom (which seems to be only a slight modification of the axiom in question)

$$F \subseteq An^2 \wedge Fnc(F) \,.\rightarrow (\exists f)(Fnc(f) \wedge F \subseteq f)$$

is consistent with respect to Zermelo-Fraenkel set theory.

The question concerning the independence of the axiom of choice is open up to now.

It is well known that Zermelo-Fraenkel and Gödel-Bernays set theories are equiconsistent. In AST the situation is not so simple. Since An plays in AST the role of ω_0 in Cantor's theory, we suppose that all formalizations of theories are parts of An . In AST we can define the notions of "formula" and "proof" either as usual (i.e. with respect to all natural numbers) or we can substitute in the usual definition the words "natural number" by the words "absolute natural number". Therefore we have in AST proofs - the length of which can be an arbitrary natural number, and absolute proofs - the length of which must be an absolute natural number. It seems better to restrict ourselves to absolute proofs. If we do not do so we can prove e.g. the following strange result:

The theory

$$AST + Con\ (ZF_{Fin}) + \neg\ Con\ (GB_{Fin})$$

is consistent with respect to Zermelo-Fraenkel (of course the length of the proof of inconsistency of GB cannot be an absolute natural number).

§ 3 Topology in AST

In this section we want to show how it is possible to define topology in AST and furtemore roughly how AST makes it possible to grasp the connection between the continuous and the discrete.

A pair $(a,\doteq)$ is called a topological space if a is a set and if $\doteq$ is an equivalence relation on it (possibly a proper class). We can interpret the relation $\doteq$ as a relation of infinitesimal nearness. First we need some definitions in which x,y denote elements of a and X,u,v denote subclasses of a .

$$Mon(x) = \{y : y \doteq x\}$$

(The monad of x is the class of all points infinitely near to x)

$$Fig(X) = \{y : (\exists x \in X)(y \doteq x)\} = \bigcup_{x \in X} Mon(x)$$

(The figure of X is the class of all points infinitely near to some point of X).

$$Sep(y,X) \equiv (\exists u,v)(Mon(y) \subseteq u \wedge Fig(X) \subseteq v \wedge u \cap v = 0)$$

(We can separate a point y from a class X if there are two disjoint sets one containing the monad of y and the other containing the figure of X).

We have the following axioms of separation:

S1 $Mon(x) \cap Mon(y) = 0 \rightarrow Sep(x, \{y\})$

S2 $Mon(x) \cap Fig(u) = 0 \rightarrow Sep(x,u)$

Of course there is the natural question as to the connection between this notion of topological space and the classical one. Now we shall define the closure operation which constitutes the classical topological space corresponding to our topological space. Let a pair $(a,\doteq)$ be a topological space and let $A \subseteq a$ be a selector with respect to $\doteq$. For every $Y \subseteq A$ we define $U(Y)$ by

$$U(Y) = \{y : y \in A \wedge \neg\ Sep(y,Y)\}$$

(the "classical closure" of Y is the class of all elements of A which cannot by separated from Y).

The class A with the closure operation U is called the skeleton of $(a,\doteq)$. For $Y,Z \subseteq A$ we have

$$U(0) = 0$$

$$U(Y \cup Z) = U(Y) \cup U(Z)$$

$$Y \subseteq U(Y)$$

and therefore the skeleton of a topological space is a classical topological space in a weak sense - Čech's closure space (see [4]). The closure of a S1-space is a semi-separated closure space since we have

$$U(\{x\}) = \{x\} .$$

If a pair $(a,\doteq)$ is a S2-space then the skeleton of it is a topological space because we have moreover

$U(U(Y)) = U(Y)$.

Constructing the skeletons we create classical topological spaces and we can ask whether we obtain enough classical topological spaces in this way. Theorem 2 gives a positive answer showing that we can obtain in this way every compact metric space.

To define the notion of metric space we need real numbers. The class of real numbers can be constructed in AST otherwise than in the classical case. As we have noted the class An plays in AST the same role as ω_0 in Cantor's theory. Therefore we define the rational numbers as pairs $\frac{n}{m}$ where $n,m \in An$ i.e.

$$Rac = \{ \pm \frac{n}{m} : n,m \in An \wedge m \neq 0 \} .$$

Moreover all pairs $\frac{n}{m}$ where n,m run over all natural numbers are called hyperrational numbers i.e.

$$HRac = \{ \pm \frac{n}{m} : n,m \in N \wedge m \neq 0 \} .$$

We define (the idea is the same as in nonstandard analysis) two hyperrational numbers x,y to be infinitely near iff their distance apart is less than $\frac{1}{n}$ for every <u>absolute</u> natural number or if both x,y are infinitely great i.e. greater than every <u>absolute</u> natural number:

$$x \doteq y \equiv (\forall n \in An)(|x - y| < \frac{1}{n} \vee (n < x \wedge n < y)) .$$

If we choose a selector with respect to $\doteq$ we obtain a class which has some of the properties of the real numbers e.g. for which the theorem about supremum holds. But there is one disadvantage - there is no x with $x^2 = 2$, we have only x with $x^2 \doteq 2$ (and similarly for the other irrational numbers). Therefore it is better to construct at first a real closed field containing HRac and to extend the equality $\doteq$ to these new elements (we add new elements to old monads e.g. $\sqrt{2}$). Then it is possible to choose a selector Real having the properties which are required from the class of real numbers (this construction is due to P. Vopěnka).

A function ρ is called hypermetric if

(1) $W(\rho) \subseteq HReal$

(2) $\rho(x,y) = 0 \equiv x = y$

(3) $\rho(x,y) = \rho(y,x)$

(4) $\rho(x,y) + \rho(y,z) \geqslant \rho(x,z) \geqslant 0$

(5) $M(\rho)$

A function ρ (possibly a proper class) satisfying (2)-(4) and

(1') $W(\rho) \subseteq Real$

(5') $(\exists b)\, \rho \subseteq b$

is called metric. A pair (a,ρ) is called hypermetric space if $D(\rho) = a^2$ and if ρ is a hypermetric(a is a <u>set</u>). A pair (A,ρ) is called metric space if $D(\rho) = A^2$ and if ρ is a metric (viz. the notion of "classical metric space", the only difference is that A,ρ need not be sets). The class of real numbers with the metric $\rho_1(x,y) = |x - y|$ becomes a metric space. Every hypermetric induces a topology if we define

$$x \doteq y \equiv (\forall n \in An)\ \rho(x,y) < \frac{1}{n}$$

We have the following metrization theorem:

<u>Theorem 1</u>.(Mlček). A topology $\doteq$ is induced by a hypermetric iff $\doteq$ is an intersection of countably many sets i.e. iff there is a class $\{d_n : n \in An\}$ such that $\doteq$ is equal to $\bigcap_{n \in An} d_n$.

This theorem has a nice history. We looked for a long time for a metrization theorem. One day J. Mlček came up with a theorem the formulation of which was rather complicated, but when P. Vopěnka formulated the above theorem, we saw that Mlček's proof worked. Therefore the theorem in question was proved before it was formulated. The names given with the following theorems indicate only the person who brought the main idea ; the other members of the seminar also participated in the creation of the results. Due to the method of work of the seminar it is very difficult to attribute a result to only one person. The proof of the last theorem was essentially simplified by K. Čuda.

<u>Theorem</u> 2. (the author). If a pair (X,ρ) is a metric space compact in the classical sense then there is a topological space

(which is moreover induced by a hypermetric space) such that its skeleton is isomorphic with (X,ρ) .

Now, we come to the crucial point of topology in AST . We shall explain the connection between the continuous and the discrete and the notion of motion. Our construction of skeletons makes it possible to view one space from two different angles and therefore to have space simultaneously discrete and continuous. The field of every hypermetric space is a set (hence formally finite) and therefore every hypermetric space is discrete, on the other hand its skeleton can be continuous (see Theorem 2).

Let a pair (a,ρ) be a hypermetric space. We call a function f (it is a set) a motion of a point if

(1) $D(f) \in N \wedge W(f) \subseteq a$

(2) $(\alpha + 1) \in D(f) \rightarrow f(\alpha) \doteq f(\alpha + 1)$

Note. The set $D(f)$ is (formally) finite, but the interesting cases are only those for which $D(f) \notin An$ holds, i.e. for which $D(f)$ is in some sense infinite.

The explanation of why we can speak about such a function as about motion, is again connected with the skeleton of the hypermetric space. For example, let (a,ρ) be a hypermetric space the skeleton of which is $(Real, \rho_1)$. Let f be the function numbering all elements between 0 and 1 and at the same time preserving the ordering (such a function exists since a itself is finite). Then, turning to the skeleton, we obtain "classical" continuous motion starting with 0 and finishing with 1.

We call a function d (it is a set) a motion of a set in the hypermetric space (a,ρ) if

(1) $D(d) \in N \wedge W(d) \subseteq P(a)$

(2) $(\alpha + 1) \in D(d) \rightarrow Mon(x) \cap d(\alpha) \hat{\approx} Mon(x) \cap d(\alpha + 1)$

(2) of the above definition demands that the cardinality of the class of all elements of $d(\alpha)$ infinitely near to x is the same as the cardinality of the class of all elements of $d(\alpha + 1)$ infinitely near to x . Therefore in $Mon(x) \cap d(\alpha)$ has n elements and $n \in An$

then $Mon(x) \cap d(\alpha + 1)$ must again have n elements. But if $Mon(x) \cap d(\alpha)$ has n elements and $n \notin An$, then $Mon(x) \cap d(\alpha + 1)$ can have m elements for every $m \notin An$ since $n \hat{\approx} m$ for every non-absolute n and m .

The definition of motion of a set is so weak that we can doubt if this definition in fact expresses the notion of "real" motion. The following theorem shows that this is so.

Theorem 3. (Vopěnka). Let d be a motion of a set in a hypermetric space. Then there is a system T of motions of points in this hypermetric space such that

(1) $\alpha \in D(d) \rightarrow d(\alpha) = \{f(\alpha) : f \in T\}$

(2) $f,g \in T \wedge \alpha \in D(d) \wedge f \neq g \,.\rightarrow f(\alpha) \neq g(\alpha)$

(3) $\alpha \in D(d) \rightarrow M(\bigcup\{f \in T : f(\alpha) \in u\})$

The first statement implies that every point of a given set $(d(0))$ has its motion in T . (2) conveys that two motions of points in T cannot go through one point. The third statement expresses the fact that the system T determines moreover the motion of every subset of $d(0)$.

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