DIFFERENTIAL CALCULUS IN THE ALTERNATIVE SET THEORY

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Motto: When we began with building up the calculus in AST we hoped to show that the Newton-Leibniz's ideas of infinitely small are possible in modern analysis, too; but now we know that the use of these ideas is even necessary.

Petr Vopěnka

In the last year's lecture (see [2]) we formulated here the axiomatic system of the Alternative Set Theory (AST) and we tried to explain some reasons which led P.Vopěnka to building up this theory (see [1]) and we proclaimed the aims we want to achieve in AST. Besides this we discussed some metamathematical aspects of this theory e.g. we proved consistency of our theory with respect to ZF set theory. Further we showed how to define basic notions of topology in AST and described a construction of the class of all real numbers (Real). At the end of the lecture we declared with Petr Vopěnka that Prague Set Seminar would start with the development of the calculus in AST during the next year. In this lecture we shall deal with some results we reached in this field. At first it is necessary to emphasize that the creation of the calculus in AST is not finished yet, although we succeded in laying the basic stones and we obtained some interesting results. For this reason it is necessary to understand this whole lecture only as a preliminary report and it is possible that some parts contained in it will still show considerable changes.

An advocate of non-standard methods will declare after my talk that we did nothing more than a reformulation of non-standard analysis in AST. And he will be right from his point of view. Nevertheless it is necessary to stress two things at least. At first the aim is to build in AST as many mathematical desciplines as possible and therefore it would be foolish to hesitate to use fruitful ideas from already created mathematical disciplines and thence non-standard ideas are used in development of mathematics in AST, too. The second and more important fact, as I think, is that our work has brought some new views to the calculus which, as I know, were not investigated in non-standard analysis at all.

Among axioms of AST (see [2]) there are all axioms of Zermelo-Fraenkel set theory for finite sets (ZF_{Fin}) i.e. all axioms of ZF set theory in which the axiom of infinity is replaced by its negation. During our investigations made in the last year we recognized that it is more convenient to strengthen these axioms at least to the axiom

2') $$V \models ZF_{Fin}$$

The difference between the previous system and the system obtained by adding 2') instead of 2) is in the fact that the first system requires induction only for meta-mathematical formulas and the strengthened one also for formal formulas. We know that the axiom 2') is independent on the others and using the other axioms we can prove that 2') is equivalent to the statement

"There is a system of classes containing V and all sets, closed under Gödel's operations such that for every class X of this system we have $(\forall x)M(X \cap x)$".

The classical analysis of Newton and Leibniz was divided into differential and integral calculi. The modern analysis (by this we mean the modern reformulation of the calculus - "$\varepsilon - \delta$ calculus") has largely obscured this difference. It is however possible to understand differential calculus as a method of concluding from the known course of a function the description of its behaviour in small neighbourhoods. Similarly integral calculus can be understoand as a method of concluding from a given description of function's behaviour in small neighbourhoods the general course of the function.

This difference between differential and integral calculi appears in AST still sharper because the course of a function - this means as the function appears to us - we can describe only on real numbers. On the other hand the description of behaviour of a function in the infinitely small neighbourhood of an investigated point requires a function defined also in points of this neighbourhood i.e. also on the points to which we extend the class of all real numbers.

I Integral calculus

Let us assume that we have a function defined on a convenient extension of Real (e.g. on H Real, see [2]) and as an example

typical for integral calculus suppose we want to know how the function describing the area determined by the given function will appear to us. This problem can be devided into two parts. At first we have to construct the function which is defined by the sums in question and in the second part we must investigate how this new function will appear to us. The first part requires therefore to master sums of non-absolute length and the second step can be solved by the method of skeletons and this method was indicated in the last year's lecture. In any case we did not hit any principal obstacle during the building up integral calculus in AST although integral calculus is not yet finished and written down.

II Differential calculus

Now we have a given function on real numbers i.e. we know how the function appears to us and we want to describe its behaviour in a differential neighbourhood. At first it is therefore necessary to extend conveniently the definition of the function into the whole differential neighbourhood and then it would be possible to define the derivative and so on. In the following we are going to deal just with extension of the function. During the last year we tested various approximations e.g. by polynomials, piecewise linear functions, continuous functions and so on and every member of our seminar has some results in this field. The problem however remained in the fact that it seemed that none approximation will retain all of the desired characteristics of the function. The following approach removes this disadvantage. Admirers of model theory will find a satisfaction in the fact that this approach is based on a theorem from model theory in AST.

In AST we can define natural numbers exactly in the same way as in Cantor's set theory and let N denotes the class of all natural numbers. Let $\alpha, \ldots$ be variables running through N. Moreover we can define the class of all absolute natural numbers An by (cf [2])

$$An = \{\alpha \in N; (\forall X \subseteq \alpha) M(X)\}$$

(a natural number is absolute if all its subclasses are sets)

Let $n, \ldots$ denote variables for absolute natural numbers. The class An is closed under arithmetical operations and we can imagine absolute natural numbers as "real" natural numbers. Moreover the class An plays in AST the same role as ω_0 in Cantor's set theory. Therefore we are going to deal with ultraproducts on An instead of ultraproducts on ω_0 as it is usual in Cantor's theory. (Although An is a proper class and therefore we cannot write An $\in$ Z. This disadvantage can be removed by a convenient coding and in the following we write An "$\in$" Z without formalism, for more details see appendix).

Theorem 1 (author). There is an isomorphism between V (with $\in$) and ultraproduct of V (with the ultraproduct relation).

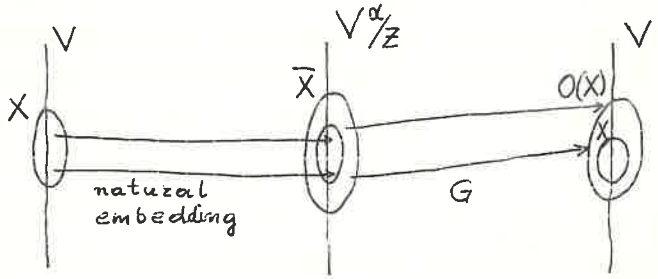
P.Vopěnka used this theorem for building up differential calculus. Let α be a fixed element of N - An, let Z be a fixed non-trivial ultrafilter on An and let G be a fixed isomorphism from the previous theorem. To every $X \subseteq V$ we put

$$\bar{X} = \{f \in {}^{\alpha}V;\ \{n;\ f(n) \in X\} \text{"}\in\text{" } Z\}$$

and we define

$$O(X) = G''\bar{X}.$$

From the well known theorem about ultraproduct (see its formulation in AST in appendix) and from the fact that G is an isomorphism we can conclude that X and O(X) satisfy exactly the same normal formulas. It is not necessary that $X \subseteq O(X)$ but classes with this property will be interesting for us and in this cases we call O(X) the standard extension of X.



P.Vopěnka proved that the class of all real numbers can be chosen in such a way that it has standard extension. J.Mlček generalized this theorem and he proved roughly speaking that every separable metric space has standard extension.

The method described above allows us to solve the problem how to extend conveniently functions since we can extend the class of all real numbers to its standard extension and moreover doing this

we simultaneously extend every function to the function which satisfies exactly the same normal formulas.

III Connection between modern and classical analysis

In the following we are going to work in AST extended by adding the operation O defined as described above. Moreover we suppose that the class Real has standard extension. We can define infinitely small numbers by $x \in O(\mathrm{Real})\ \&\ (\forall n \in An)(|x| < \frac{1}{n})$ and hence in this theory we are able to realize differential calculus of Newton and Leibniz with infinitely small numbers. On the other hand all formulas of modern analysis can be expressed in our theory by formulas in which the operation O does not occur. There arises therefore a very natural question whether when using formulas with the operation in question we can express more properties than we can describe using only formulas in which this operation does not occur. In other words we ask if one can express in Newton-Leibniz's analysis more than in modern analysis. In this direction we have the following result (d_i being variables for infinitely small numbers, (0,x) denoting the interval between 0 and x).

Metatheorem 2 (P.Vopěnka). Let φ be a normal formula then in AST is provable

$$(\exists d_1)(\forall d_2)\varphi(d_1,d_2,x_1,\dots,x_k,O(X_1),\dots,O(X_m)) \equiv$$
$$\equiv (\exists n_2)(\forall n_1)(\exists y_1 \in (0,\tfrac{1}{n_1}))(\forall y_2 \in (0,\tfrac{1}{n_2}))\varphi(y_1,y_2,x_1,\dots,x_k,O(X_1),$$
$$,\dots,O(X_m)) \equiv (\exists n_2)(\forall n_1)(\exists y_1 \in (0,\tfrac{1}{n_1}))(\forall y_2 \in (0,\tfrac{1}{n_2}))\varphi(y_1,y_2,x_1,$$
$$,\dots,x_k,X_1,\dots,X_m).$$

For example we define using the notion of infinitely small numbers that a function F is continuous if

$$(\forall d_1)(\exists d_2)(F(x+d_1) = F(x)+d_2)$$

and therefore this notion can be reformulated by previous statement to

$$(\forall n_2)(\exists n_1)(\forall y_1 \in (0,\tfrac{1}{n_1}))(\exists y_2 \in (0,\tfrac{1}{n_2}))(F(x+y_1) = F(x)+y_2)$$

i.e. to the formula

$$(\forall n_2)(\exists n_1)(\forall y_1 \in (0,\tfrac{1}{n_1}))(|F(x+y_1)-F(x)| < \tfrac{1}{n_2})$$

and this is exactly the formulation used in modern analysis.

One can ask whether Vopěnka's metatheorem works also for formulas having three quantifiers for infinite small numbers i.e. if

$$(\forall d_1)(\exists d_2)(\forall d_3)\varphi \equiv$$
$$\equiv (\exists n_3)(\forall n_2)(\exists n_1)(\forall y_1 \in (0,\tfrac{1}{n_1}))(\exists y_2 \in (0,\tfrac{1}{n_2}))(\forall y_3 \in (0,\tfrac{1}{n_3}))\varphi$$

holds for every normal formula φ . K.Čuda constructed a normal formula for which this is not right (see [3]). Although Čudaś contraexample shows that Vopěnka's theorem cannot be generalized, Čuda's formula can be expressed by a formula in which variables for infinitely small (and operation O, too) do not occur. But this is not the case for all formulas since we can prove (F being variable for real function).

Metatheorem 3 (author). There is a formula $\varphi(F)$ (of the type $(\exists k \in O(An))(\forall r \in Real)\tilde{\varphi}(O(F),k,r)$ where $\tilde{\varphi}$ is a normal formula) such that there is no normal formula $\phi(F)$ (with parameters An and Real, say) in which the operation O does not occur and with

$$AST \vdash \varphi(F) \equiv \phi(F)$$

We can therefore say that in a very natural sense it is possible to express in Newton-Leibniz 's analysis more properties than in modern analysis (modern reformulation of the calculus). If one wants to be invidious one can say that modern analysis is only a part of classical analysis. Very serious investigation must now show how large this part is. This part may contain everything interesting but it is also possible that there are very important branches of Newton-Leibniz 's analysis which cannot be described in modern analysis at all.

Appendix.

The aim of this part is to prove Theorem 1 and Metatheorem 3. Doing this we have to define ultraproduct in AST and we shall prove some facts about ultrafilters. We are going to restrict ourselves to ultrafilters on An since in the general case there are problems how to define the class $\bar{X}$ to every X. The definition of ultraproduct is very similar to the usual one in Cantor's set theory although we shall essentially use the axiom of extension (axiom (4) [2]) to prove some theorems about ultraproduct. In the following we are not going to prove theorems which have similar proofs as theorems of

Cantor's theory but we only refer to proofs in question.

At first we need definition of formal formula. Since the class An plays in AST the same role as ω_0 plays in Cantor's set theory, we define the notion of "formula" substituting in the usual definition the words "natural number" by the words "absolute natural number". Then formulas are absolutely finite sets or if we wish absolute natural numbers. We define the satisfaction relation for every model (possibly determined by proper classes) by the usual definition (we can prove that there is exactly one satisfaction relation since every part of An has the first element). We shall deal with models with absolute equality.

DEFINITION. A model $\mathfrak{A}$ is called saturated if for every sequence of formulas $\{\varphi_i(x);\ i \in An\}$ with parameters in the field of $\mathfrak{A}$ we have

$$(\forall i \in An)(\exists a)(\mathfrak{A} \models \bigwedge_{j \leq i} \varphi_j[a]) \rightarrow (\exists a)(\forall i \in An)\ \mathfrak{A} \models \varphi_i[a]$$

LEMMA. There is no countable saturated model.

Proof: see the standard proof of this lemma in Cantor's theory.

THEOREM. Let $\mathfrak{A}$, $\mathfrak{B}$ be saturated models which are elementarily equivalent. Then there is an isomorphism between them.

Proof. By the axiom of cardinalities (axiom (6) [2]) we have only one uncountable cardinality. Hence every two saturated elementarily equivalent models have the same cardinality by the previous Lemma. Therefore the standard proof that two elementarily equivalent models of the same cardinality are isomorphic works.

THEOREM. $< V,E >$ is a saturated model.

Proof. Suppose $< V,E > \models \bigwedge_{j \leq i} \varphi_j[a_i]$ for every $i \in An$. By the axiom of extension there is a set function f such that $f(i) = a_i$ for every $i \in An$. Put $b = W(f)$ and $Y_i = \{x;\ x \in b\ \&\ < V,E > \models \varphi_i[x]\}$. By the equivalent form of the axiom 2'), Y_i is a set for every $i \in An$. There is a set function g with $(\forall i \in An)g(i) = Y_i$ again by the axiom of extension. If we define $Y = \{\gamma \in D(g);\ \bigcap_{\beta \subseteq \gamma} g(\beta) \neq 0\}$

we have $An \subseteq Y$ since $a_i \in \bigcap_{j \leq i} Y_j$. In the definition of Y we use only sets as parameters and therefore $M(Y)$ since all ZF_{Fin} axioms hold in AST. The class An itself is not a set and from this $An \neq Y$ follows. Let $\gamma \in Y - An$ and $a \in \bigcap_{\beta \leq \gamma} g(\beta) \subseteq \bigcap_{j \in An} Y_j$ then for every $i \in An$ we have $\langle V,E \rangle \models \varphi_i[a]$ by the definition of Y_i.

For the following let us fix $\alpha \in N - An$.

DEFINITION. A class Z is called a (non-trivial) ultrafilter on An (in symbols ULTR(Z)) if $Z \subseteq P(\alpha)$ and

(a) $(\forall x,y \in P(\alpha))(An \subseteq x \cup y \rightarrow (x \in Z \vee y \in Z))$

(b) $(\forall x,y \in Z)(x \cap y \in Z)$

(c) $0 \notin Z \ \& \ (\forall n \in An)(\{n\} \notin Z)$

LEMMA. If Z is an ultrafilter on An then

(a) $(\forall x,y \in P(\alpha))(x \cap An = y \cap An \rightarrow (x \in Z \equiv y \in Z))$

(b) $(\forall x,y \in P(\alpha))((An \subseteq x \cup y \ \& \ x \cap y = 0) \rightarrow (x \in Z \equiv y \notin Z))$

(c) $(\forall x,y \in P(\alpha))((x \in Z \ \& \ y \in Z) \equiv x \cap y \in Z)$

Proof. Let $y \cap An \subseteq x \ \& \ y \in Z$, then $\alpha \supseteq (x \cup (\alpha - y)) \supseteq An$ and hence $x \in Z \vee (\alpha - y) \in Z$ by (a) from the definition of ultrafilter. If $(\alpha - y) \in Z$ then $0 = (\alpha - y) \cap y \in Z$ which is a contradiction with (c) from the definition of ultrafilter. Therefore we have proved the statement (a), the others are trivial.

THEOREM. There is no coding of all ultrafilters on An by sets i.e. $(\forall R)(\exists Z)(ULTR(Z) \ \& \ (\forall x)(Z \neq R''\{x\}))$.

Proof can be done as it is proved in Cantor's set theory that there is $2^{2^{\aleph_0}}$ ultrafilters on ω_0.

Let $\mathfrak{A} = \langle A,R \rangle$ be a model (with one relation, say) and let Z be an ultrafilter on An. Put

$$A_1 = \{f; D(f) = \alpha \ \& \ f''An \subseteq A\}.$$

Let us define analogically as in Cantor's theory

$$f =_{Ul} g \equiv (f,g \in A_1 \ \& \ (\exists u \in Z)(u \cap An = \{n;\ f(n) = g(n)\})).$$

Then $=_{Ul}$ is an equivalence and thence we can using the axiom of choice (axiom(5) [2]) choose a selector of this equivalence. Let us denote the chosen selector by A^{Ul} and define

$$R^{Ul} = \{\langle f,g \rangle;\ f,g \in A^{Ul} \ \& \ (\exists u \in Z)(u \cap An = \{n; \langle f(n),g(n) \rangle \in R\}\}$$

The model determined by A^{Ul} and R^{Ul} is called ultraproduct and denoted by $Ul(\mathfrak{A},Z)$. Analogically as in Cantor's set theory we can prove the following theorems although we need the axiom of extension in these proofs. (Exactly the same trick as has to be used here is used in the following metatheorem).

THEOREM. $Ul(\mathfrak{A},Z) \models \varphi(f_1,\dots,f_k)$ iff $(\exists u \in Z)(u \cap An = \{n;\ \mathfrak{A} \models \varphi(f_1(n),\dots,f_k(n))\})$.

THEOREM. $Ul(\mathfrak{A},Z)$ is a saturated model.

In the special case that $\mathfrak{A} = \langle V,E \rangle$ we obtain Theorem 1 by the combination of previous statements. Moreover in this case we define

$$\bar{X} = \{f \in V^{Ul};\ (\exists u \in Z)(u \cap An = \{n;\ f(n) \in X\})\}.$$

Now we want to show that for every normal formula φ we have $\varphi(X_1,\dots,X_k) \equiv \varphi(\bar{X}_1,\dots,\bar{X}_k)$. If $f \in V^{Ul}$ then we put $\hat{f} = E^{Ul}{}''\{f\}$. Let X^* be variable for parts of V^{Ul} and let us define

$$X^* \in^* Y^* \equiv (\exists f \in Y^*)(X^* = \hat{f})$$

Note. Classes $\bar{X}$ are parts of V^{Ul} and therefore it is convenient to speak about $\hat{f}$ (part of V^{Ul}) rather than about f. Of course we have for every set formula φ,

$$AST \vdash (Ul(\langle V,E \rangle,Z) \models \varphi(f_1,\dots,f_k)) \equiv \varphi^*(\hat{f}_1,\dots,\hat{f}_k)$$

Metatheorem. Let φ be a normal formula. Then in AST is provable

$$\varphi^*(\hat{f}_1,\dots,\hat{f}_k,\dots,\bar{X}_1,\dots,\bar{X}_m) \equiv (\exists u \in Z)(u \cap An = \{n;\ \varphi(f_1(n),\dots,f_k(n),X_1,\dots,X_m)\}).$$

Demonstration. We have $\hat{f} = \hat{g} \equiv E^{Ul}{}''\{f\} = E^{Ul}{}''\{g\} \equiv f =_{Ul} g \equiv f = g$ by one of the previous theorems and for V^{Ul} is a selector with respect to $=_{Ul}$. Let $f \in X^* - Y^*$ than $\hat{f} \in^* X^* \ \& \ \hat{f} \notin^* Y^*$ and hence $X^* = Y^* \equiv X^* =^* Y^*$. This enables us to restrict ourselves to the following two kinds of atomic formulas:

(a) $\hat{f} \in^* \hat{g} \equiv f \in \hat{g} \equiv \langle f,g \rangle \in E^{Ul} \equiv (\exists u \in Z)(u \cap An = \{n;\ f(n) \in g(n)\})$

(b) $\hat{f} \in^* \bar{X} \equiv f \in \bar{X} \equiv (\exists u \in Z)(u \cap An = \{n;\ f(n) \in X\})$

The induction steps for &, ¬ and the step showing $(\exists f)\varphi^*(\hat{f},\hat{f}_1,\ldots,\hat{f}_k,\bar{X}_1,\ldots,\bar{X}_m) \rightarrow (\exists u \in Z)(u \cap An = \{n;\ (\exists x)\varphi(x,f_1(n),\ldots,f_k(n),X_1,\ldots,X_m)\})$ can be proved as usual. Let $u \in Z$ & $u \cap An = \{n;\ (\exists x)\varphi(x,f_1(n),\ldots,f_k(n),X_1,\ldots,X_m)\}$. Then we can choose a function F with $u \cap An = \{n;\varphi(F(n),f_1(n),\ldots,f_k(n),X_1,\ldots,X_m)\}$. By the axiom of extension there is a set function f with $F \subseteq f$ and we can suppose that $f \in V^{Ul}$. By the induction hypothesis we have $\varphi^*(\hat{f},\hat{f}_1,\ldots,\hat{f}_k,\bar{X}_1,\ldots,\bar{X}_m)$.

Now we intend to prove Metatheorem 3. To make the demonstration clear we devide it into five parts.

1) For every ultrafilter Z_2 there is an ultrafilter Z_1 with

$$(\forall f)(\exists v \in Z_1)(\forall u \in Z_2)(u \cap An \neq \{n; f(n) \in v \cap An\}).$$

(Cf. Rudin-Keisler's ordering)

To prove this statement fix an ultrafilter Z_2 and define

$$R = \{\langle v,f\rangle ; (\exists u \in Z_2)(u \cap An = \{n; f(n) \in v \cap An\})\ \&\ v \subseteq \alpha\}\ .$$

Since we cannot code all ultrafilters by sets we can fix an ultrafilter Z_1 with $(\forall f)(R''\{f\} \neq Z_1)$. Let f be arbitrary function and suppose first $Z_1 - R''\{f\} \ni v$. Then $v \in Z_1$ and $(\forall u \in Z_2(u \cap An \neq \{n;\ f(n) \in v \cap An\})$. The second possibility is that $R''\{f\} - Z_1 \ni w$ i.e. $(\exists \tilde{u} \in Z_2)(\tilde{u} \cap An = \{n;\ f(n) \in w \cap An\})\ \&\ (-w) \in Z_1$. If there is $u \in Z_2$ such that $u \cap An = \{n;\ f(n) \in (-w) \cap An\}$ then $0 \neq \tilde{u} \cap u \cap An = \{n;\ f(n) \in w \cap (-w) \cap An\}$ which is a contradiction. Hence we have proved $(\exists v \in Z_1)(\forall u \in Z_2)(u \cap An \neq \{n;\ f(n) \in v \cap An\})$ and therefore the required property again holds for f we have started with.

2) Fix Z_1, Z_2 with the above described property. Z_1 is not countable and hence by the axiom of cardinalities there must be a 1-1 mapping Q such that $Q''(1,2) = Z_1$. Let us define

$$H(r \cdot \tfrac{1}{2^k}) = 1 \quad \text{iff} \quad k \in Q(r)$$

(i.e. we code members of Z_1 by real numbers from the interval (1,2) and we define $H(r \cdot \frac{1}{2^k}) = 1$ iff k is a member of the element of Z_1 the code of which is r)

3) Define $\varphi(F)$ by $(\exists k \in O(An))(\forall r \in Real)(1 < r < 2 \rightarrow O(F)(r \cdot \frac{1}{2^k}) = 1)$.

We have two operations of "closure" $\bar{X}^1$ and $\bar{X}^2$ according to ultrafilters Z_1 and Z_2. Moreover using ultraproducts Ul $(\langle V,E\rangle, Z_1)$ and Ul $(\langle V,E\rangle, Z_2)$ we define two notions of standard extension O_1 and O_2 and let $\varphi_i(F)$ be the formula which we obtain by substitution of O_i for O in the formula $\varphi(F)$. In the following two steps we prove that $\varphi_1(H)$ is true and that $\varphi_2(H)$ does not hold. This finishes the demonstration since if we have a normal formula in which the operation O does not occur with AST $\vdash \varphi(F) \equiv \psi(F)$ it would be provable in AST that $\varphi_1(F) \equiv \psi(F) \equiv \varphi_2(F)$ and this would be a contradiction.

4) $\varphi_i(H) \equiv$

$$\equiv (\exists k \in O(An))(\forall r \in Real)(1 < r < 2 \rightarrow O(H)(r \cdot \tfrac{1}{2^k}) = 1) \equiv$$

$$\equiv (\exists f \in \bar{An}^i)(\forall r \in Real)(\bar{1}^i < \bar{r}^i < \bar{2}^i \rightarrow \bar{H}^i(\bar{r}^i \cdot \tfrac{1}{(\bar{2}^i)^{\hat{f}}}) = \bar{1}^i)^* \equiv$$

$$\equiv (\exists f \in \bar{An}^i)(\forall r \in Real)(\exists u \in Z_i)(u \cap An = \{n;\ 1 < r < 2 \rightarrow H(r \cdot \tfrac{1}{2^{f(n)}}) = 1\}) \equiv$$

$$\equiv (\exists f \in \bar{An}^i)(\forall r \in Real)(\exists u \in Z_i)(u \cap An = \{n;\ f(n) \in Q(r)\}) \equiv$$

$$\equiv (\exists f \in \bar{An}^i)(\forall v \in Z_1)(\exists u \in Z_i)(u \cap An = \{n;\ f(n) \in v\})$$

5) We have evidently $\neg\varphi_2(H)$ by the choice of ultrafilters Z_1 and Z_2. Let d be the diagonal (i.e. $d = \{\langle\beta,\beta\rangle; \beta \in \alpha\}$ then $\{n;\ d(n) \in An\} = An$ and therefore $d \in \bar{An}^1$ and moreover we have trivially $(\forall v \in Z_1)(\exists u \in Z_1)(u \cap An = \{n;\ d(n) \in v\})$ and hence we are done.

Using the ideas from the previous demonstration we can construct a formula in the language of non-standard analysis (this formula has one bounded variable for elements of Real and one bounded variable for elements of Real* such that this property of real function cannot be expressed by a formula quantifiing only variables for Real. But we have to assume that we deal with all non-standard models. The problem is whether this result works if we consider only models which are enlargements with respect to Real[1]. On the other hand we can strengthen our result supposing e.g. that the ultrafilters Z_1 and Z_2 are "selective" and ultraproducts are elementarily equivalent in "nonstandard language.

[1] The problem is solved.

REFERENCES:

[1] P.Vopěnka, Matematika v alternativni teorii množin (Mathematics in the Alternative Set Theory) manuscript

[2] A.Sochor, The Alternative Set Theory, Set Theory and Hierarchy Theory A Memorial Tribute to A.Mostowski, Lecture Notes in Math.537 p.259-271

[3] K.Čuda, The relation between ε – δ procedures and infinitely small in nonstandard models, this Proceedings

THE CONSISTENCY OF THE THEORY $ZF + L^1 \neq HOD$

by

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In [3] J. Myhill and D. Scott proved that $ZFC \vdash L^1 = HOD$. S. Roguski [4] proves that every model M of ZFC can be the class HOD of some generic extension of M. Hence the theories of the classes HOD and L^1 are equal over ZF i.e. $ZF \vdash \Phi^{L^1} \longleftrightarrow ZFC \vdash \Phi \longleftrightarrow ZF \vdash \Phi^{HOD}$ for any sentence Φ.
Nevertheless $L^1 = HOD$ cannot be proved in ZF only. We show the following

THEOREM 0. If ZF is consistent then $ZF + L^1 \neq HOD$ is consistent too.

First let us recall the definition of the class L^1

$$L^1_\lambda = \bigcup_{\xi<\lambda} L^1_\xi \quad \text{for limit } \lambda$$

$L^1_{\alpha+1}$ = the family of all subsets of L^1_α definable by second order formulas with parameters from L^1_α.

The proof is based on the observation that L^1 depends on subsets of On only, i.e. any two models of ZF with the same sets of ordinals have the same classes L^1.

DEFINITION 1.

$$\mathcal{P}On = \bigcup_\alpha \mathcal{P}(\alpha)$$

$$L^* = L[\mathcal{P}On] = \bigcup_\alpha L[\mathcal{P}(\alpha)] \ :$$

LEMMA 2. $L^1 = (L^1)^{L^*}$.

Proof. We prove that for every α

(*) $L^1_\alpha = (L^1_\alpha)^{L^*}$.

Assume that (*) holds for some α. Then there is a 1-1 function $f \in (L^1)^{L^*} \subseteq L^*$, $f : L^1_\alpha \to On$. Each $u \subseteq L^1_\alpha$ can be coded as