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Mathematics in the Alternative Set Theory

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The decisive part of contemporary mathematics can be characterized as mathematics in Cantor set theory. Main principles of Cantor set theory are derived from the notion of actually infinite sets; thus contemporary infinitary mathematics studies actual infinity. Alternative set theory studies infinity as a phenomenon involved in our observation of large, incomprehensible sets. Hence properties of infinity in Alternative set theory differ from properties of infinity in Cantor set theory. This book indicates how mathematics can be developed in Alternative set theory. From the formal and technical point of view, Alternative set theory is rather near to Non-standard Analysis and can be considered, from this point of view, a particular case of Non-standard Analysis.

Den wesentlichen Teil der gegenwärtigen Mathematik kann man als eine Mathematik charakterisieren, die sich in der Cantorschen Mengenlehre darstellen läßt. Die Hauptprinzipien dieser Mengenlehre sind aus dem Begriff der aktual unendlichen Mengen abgeleitet; daher studiert die gegenwärtige infinitäre Mathematik das Aktual-Unendliche. Die Alternative Mengenlehre untersucht das Unendliche als eine Erscheinung, die unsere Erfahrung mit großen, unüberschaubaren Mengen enthält. Daher unterscheiden sich die Eigenschaften des Unendlichen in der Alternativen Mengenlehre von den Eigenschaften des Unendlichen in der Cantorschen Mengenlehre. Dieses Buch zeigt, wie die Mathematik aus der Alternativen Mengenlehre entwickelt werden kann. Vom formalen und technischen Standpunkt aus ähnelt die Alternative Mengenlehre sehr der Nichtstandard-Analysis und kann, unter Berücksichtigung dieses Standpunktes, als spezieller Fall der Nichtstandard-Analysis aufgefaßt werden.

L'essentiel des mathématiques contemporaines peut être caractérisé comme la partie des mathematiques qu'il est possible d'exposer a partir de la théorie cantorienne des ensembles. Puisque les principes fondamentaux de la théorie cantorienne des ensembles sont dérivés de la notion d'ensemble actuellement infini, les mathématiques modernes étudient l'infini actuel.

En théorie alternative des ensembles on étudie l'infini comme un phénomène inhérent à notre expérience des grands ensembles impossibles a reduire. C'est pourquoi les propriétés de l'infini en théorie alternative des ensembles sont différentes de celles en théorie de Cantor. Dans ce livre, on montre comment les mathématiques peuvent être développées a partir de la théorie alternative des ensembles. Du point de vue formel et du point de vue technique, la théorie alternative des ensembles ressemble à l'analyse non-standard et de ce point de vue peut être considérée comme un cas spécial d'analyse non-standard.

Современную математику, или, точнее, ее существенную часть, можно охарактеризовать как математику в Канторовой теории множеств. Основные принципы Канторовой теории множеств выведены из представления актуально бесконечных множеств, и таким образом современная бесконечная математика изучает актуальную бесконечность.

Альтернативная теория множеств изучает бесконечность как явление, сопровождающее наше наблюдение больших, необозримых множеств. Поэтому свойства бесконечности в альтернатвной теории множеств другие, чем в теории Кантора. В работе указанно, каким образом можно строить математику в альтернативной теории множеств. С формально технической точки зрения, альтернативная теория множеств весма близка нестандартному анализу. В этом смысле ее можно считать частным случаем последнего.

Preface

The main principles of the Alternative Set Theory and of mathematics based on it were formulated by the present author in 1973. Since then the Alternative Set Theory has been developed in a seminar at Charles University, Prague, headed by the author. I am very much indebted to members of this seminar. First of all, A. Sochor contributed considerably to the development of the theory and investigated metamathematical questions concerning it. Also K. Čuda and J. Mlček made valuable contributions.

The present work contains only topics elaborated on by the author in a more or less complete form. Thus, the author's isolated special results and topics investigated by the above named colleagues have been eliminated as well as topics whose treatment is not fully in the spirit of the Alternative Set Theory. Only those results of my colleagues have been included that are indispensable for our exposition.

Several results not included here will be published in a series of papers by various authors; final versions of these papers are being prepared. The papers will show in greater detail how various branches of mathematics should be developed in the Alternative Set Theory.

Intuitive exposition and questions of conception are emphasized. The technical parts serve as illustrations of how methods usual in Cantor Set Theory, Nonstandard Analysis etc. can be imitated the Alternative Set Theory.

From the formal and technical point of view, Alternative Set Theory is rather near to Nonstandard Analysis and can be, considered, from this point of view, for a particular case of Nonstandard Analysis.

My colleague J. Polívka helped to improve various motivations. The author's students M. Resl and A. Vencovská contributed several minor comments.

Finally, the author wishes to thank his friend P. Hájek who translated the whole text into English and to P. Hinman who proofread the English text of Chapter I - V. They both helped by various comments to improve the text.

There were several preliminary lectures on Alternative Set Theory see e.q. A. Sochor's papers.

Contents

Introduction

1. Cantor set theory

Cantor set theory is a mathematical theory of finite and actually infinite sets. It canonizes the main principles accepted by mathematicians as true assertions about sets.

Fundamentals of the theory of finite and infinite sets are due to B. Bolzano, who formulated some of its principles. Later G. Cantor developed set theory in a systematic manner and extended it by some further principles.

Some of the principles forming the basis of set theory concern finite sets and have been accepted since the beginnings of civilization as basic truths not only by mathematicians but by all people. Due to our education and to the verifiability of very special cases occurring in everyday practice, these principles are accepted as self-evident. But a critical analysis shows a lack of arguments for their absolute acceptance. For example, we are convinced that all countings of a given finite set by natural numbers yield the same number of elements. If we tried to prove this assertion by mathematical induction, we would only reduce our problem to the analogous question of the truthfulness of mathematical induction. Imagine that a set could be found for which one counting gives n elements and another gives m elements, where m and n are two particular quite concrete unequal numbers, or, at least, that reasons for the existence of such a set could be exhibited. On the other hand, assume that the statement saying that each set having n elements by one counting has n elements by any other counting has been proved by a concrete proof. Such a proof would be realized according to the instruction given by mathematical induction. The situation described would lead to the conclusion that proofs obeying laws of logic have decreasing convincingness as their length increases. From this we could conclude that even finitary mathematical statements can have quite complicated truth values, not just two, and so on.

But the main subject of Cantor set theory is infinite sets; their existence is assumed. The main postulates of Cantor set theory concerning infinite sets were not formulated at random, and they are not immediately derivable from the mere assumption that an actually infinite set exists. Their motivation can be found in mathematics long before the origin of set theory. This concerns mainly the following two types of postulates.

Pre-set-theoretical mathematics investigated objects of several precisely specified sorts, admitting unlimited construction of objects of a given sort as, e.g., natural numbers, real numbers, points in the space and so on. The infinity inherent in such sorts of objects is <u>potential</u> infinity. The first kind of postulate of Cantor set theory is motivated by the treatment of this infinity as actual, i.e. by the declaration that all objects capable of successive construction have already been constructed. Repeated generalization and higher precision have resulted in the determination of the form of properties of objects so that the collection of all objects having such a property is understood as an actually existing set. At the same time it has become apparent that there are limits of generalization in this direction.

In addition to objects of a certain sort, pre-set-mathematics dealt with objects being in one-one correspondence with subcollections of the set of all objects of a certain sort. To give an example, straight lines, planes etc. were also studied in addition to points in space. Thus we can form the set of all straight lines for a set of certain subcollections of the set of all points. As the ultimate generalization of this approach one obtains the postulate guaranteeing the existence of the set of <u>all</u> subsets of a given set. In contradistinction to postulates of the first kind, the postulate of power-set were apparently not known to Bolzano.

At present the existence of actually infinite sets has become a dogma believed in by most mathematicians; moreover, mathematicians try to implant it in the minds of other people. At the same time, we are unable to give evidence of any actually infinite set in the real world. Thus we deal here with a construction extending the real world and surpassing qualitatively the limits of the space of possibilities of our observation. Assertions about infinite sets thus lose their phenomenal content. The result is that the further development of set theory has become entirely dependent on formal means, which are the only sure guide in the darkness sets are clouded in.

This fact caused difficulties in the very beginnings of set theory. It appeared that the natural postulates of Cantor set theory were insufficient for the decision of the question of the truth of the axiom of choice. At the same time, the question was so urgent that it was impossible to wait for formal confirmation of independence of this axiom. There was no motivation analogous to that of previous postulates. Eventually, general acceptance of the axiom of choice was decided for purely formal reasons; this axiom simplifies considerably the structure of infinite sets and yields several very elegant theorems. Attempts to motivate the axiom of choice by its validity for finite sets were disqualified by the axiom of determinateness, which can also be motivated by its validity for finite sets but which is inconsistent with the axiom of choice. Moreover, it is possible that there are various other axioms of this kind. It suffices to take any statement true for finite sets, acceptable or even technically advantageous for infinite sets, but inconsistent with the axiom of choice.

Today a considerable number of sentences of set theory independent (neither provable nor refutable) from the basic axioms of set theory is known. Mathematicians are looking without success for further principles strong enough to decide their truth. The continuum hypothesis is a typical example. Acceptance of the continuum hypothesis brings some technical advantages, but set theory with the continuum hypothesis negated is rather interesting. Thus we have no unique set theory; instead, we have various set theories for which the original Cantor set theory serves as their common frame.

Moreover, it is possible to formulate different postulates for actually infinite sets and create in this way a theory of actually infinite sets distinct from Cantor set theory. For example, the power set axiom can be replaced by the postulate saying that each infinite set can be bijectively mapped onto the set of all natural numbers. The resulting theory could probably compete quite succesfully with Cantor set theory.

Efforts of mathematicians to fully grasp actual infinity have been unsuccessful. But this does not diminish the importance of Cantor set theory, which remains a document of human aspirations to surpass limits of space in a way having no analogy in history.

2. Mathematics in Cantor set theory

The significance of Cantor set theory for mathematics does not lie only in the theory itself but also in its position within mathematics. Soon after the origin of set theory it became clear that it was useful mainly for the following three reasons.

All mathematical objects created in pre-set-mathematics can be reconstructed as structures in set theory. More precisely, those objects were given in set theory their canonical models so that one could replace investigation of the original objects by investigation of their respective models. In some cases this replacement influenced the original notions and caused a modification in accordance with the model. Real numbers, infinitesimal calculus etc. may be taken as examples.

In pre-set-mathematics there occurred infinities of various kinds. For example, infinity as the unlimited possibility of constructing objects of a certain sort, infinity as an unboundedly increasing quantity, infinity where two parallels meet, etc. All of these kinds of infinity were reduced to actual infinity dealt with in set theory. Set theory became a general theory of infinity.

Set theory gave to mathematics the combinatorially richest structure, namely the structure of finite and actually infinite sets. This caused the origin of new mathematical disciplines. Some of them directly use the structure of sets, at least partly, such as topology, measure theory, etc. Some others present the superstructure of classical structures as for example functional analysis and some algebraical structures etc. Furthermore, set theory offers an inexhaustible variety of different abstract structures.

Cantor set theory thus became the world of all mathematics, i.e. the place where the whole of mathematics is located. Particular mathematical disciplines were deprived of the responsibility for their consistency, since this responsibility was relegated to set theory.

But this conception of mathematics has considerable disadvantages that are increasingly attracting the attention of mathematicians.

Some disciplines pursued in pre-set-mathematics had to be considerably violated in order to include them into set theory. Moreover, some approaches to their construction were absolutized. Infinitesimal calculus serves as an example. At the time of its reconstruction set theory was not developed enough to make the most natural modelling possible. This is because some of the leading ideas of infinitesimal calculus were disqualified and its subject was extended in an inadequate manner. Immediate calculations were replaced by proofs often hiding substantial ideas. Moreover, this so-called ε, δ-analysis does not reflect perfectly the original infinitesimal calculus since some notions in which infinitely small quantities are quantified are not translatable into ε, δ-analysis and must be eliminated when using functional analysis.

Set theory brought a whole scale of particular cases of actual infinity into mathematics. But most of them cannot be reasonably interpreted in the real world. Their existence is a mere consequence of the basic conception of actual infinity in Cantor set theory.

Set theory opened the way to the study of an immense number of various structures and to an unprecedented growth of knowledge about them. This caused a scattering of mathematics. Moreover, most results of this kind derive their sense only from the existence of the respective structure in Cantor set theory. Mathematics based on Cantor set theory changed to mathematics of Cantor set theory.

Cantor set theory is responsible for this detrimental growth of mathematics; on the other hand, it imposed limits for mathematics that cannot be surpassed easily. All structures studied by mathematics are a priori completed and rigid, and the mathematician's role is merely that of an observer describing them. This is why mathematicians are so helpless in grasping essentially inexact things such as realizability, the relation of continuous and discrete, and so on.

Contemporary mathematics thus studies a construction whose relation to the real world is at least problematic. Moreover, this construction is not the only possible one and, as a matter of fact, it is not the most suitable from the point of view of mathematics itself. This makes the role of mathematics as a scientific and useful method rather questionable. Mathematics can be degraded to a mere game played in some specific artificial world. This is not a danger for mathematics in the future but an immediate crisis of contemporary mathematics. It manifests itself in the fact that most quite deep, even ingenious mathematical results are entirely uninteresting not only for people who are not mathematical professionals but even for other mathematicians at present working on problems with differently situated pieces on the chessboard.

Some mathematicians recognizing this crisis react by forbidding or at least discrediting work in some mathematical disciplines. Needless to say, such an attitude is undignified. Mathematics certainly cannot be mechanically expurgated.

3. The alternative set theory

One possible way out of the crisis of contemporary mathematics may be through an attempt to reconstruct mathematics on a phenomenal basis. This should result in a natural elimination of artificial problems from mathematics, just as investigations of the figure consisting of a half-circle together with a parabola were eliminated from classical geometry.

But a purely phenomenal conception of mathematics would considerably impoverish mathematics; moreover, this impoverishment would affect the role of mathematics itself. Mathematics is a means for surpassing the horizon of human experience. We use mathematics to express thoughts preceding our knowledge and for which later evidence is often impossible to obtain. In reconstructing mathematics we are thus obliged to accept also basic principles for surpassing the horizon of evidence. In particular, we do not reject logic as a means of deduction from axioms; but we shall also present some criticism of logic.

Due to the central role of set theory in mathematics it seems reasonable to begin by presenting a new set theory as a possible basis for mathematics, just as contemporary mathematics is based on Cantor set theory. This book is an attempt to create such an alternative set theory.

We shall deal with the phenomenon of infinity in accordance with our experience, i.e., as a phenomenon involved in the observation of large, incomprehensible sets. We shall by no means use any ideas of actually infinite sets. Let us note that by eliminating actually infinite sets we do not deprive mathematics of the possibility of describing actually infinite sets sufficiently well in the case that they would prove to be useful. Moreover, we shall be able to offer several theories that could be used for theories of actual infinity, one of them being Cantor set theory.

The careful reader will realize that our theory is not even fully based on the classical concept of finite sets. This is only implicit in the present book since systematic development in this direction would make the book considerably longer and would stress topics not suitable for a first acquaintance with our theory. We have the same reason for the development of our theory inside classical logic.

Motivations of certain approaches in our theory remind one of the ideas of A. S. Esenin-Volpin; but the author feels unable to present a more detailed analysis of these relations.

We develop the alternative set theory at a time when Cantor set theory has been considerably elaborated. This enables us to take over various techniques and results almost literally from Cantor set theory; they will only be interpreted differently. Let us illustrate this by an example. In a popular booklet the author tries to elucidate properties of countable sets. He invites the reader into a hotel having an infinite number of rooms enumerated by natural numbers; all rooms are taken. In spite of this, it is possible to accept a new guest, accommodating him in Room 1 and at the same time move each guest from Room n to Room (n+1). Now imagine that our hotel has only one thousand rooms, all taken. We can do the same. The new guest is accommodated in Room 1, the guest from Room 1 is moved to Room 2 etc. Since guests are moved successively the process will not be finished during one day and, analogously as above, each guest will be accommodated for almost the whole day. In this case the set of thousand rooms contains a subcollection (the subcollection of all rooms to which guests are potentially moved) that behaves somewhat like a countable set in Cantor set theory.

Like Cantor set theory, our theory is a non-formalized "naive" theory. Nevertheless, some of its important fragments can be axiomatized. Thus for example almost all results contained in the present book are deduced from axioms that are explicitly stated. The formal theory obtained in this way has simple models in any of the axiomatic systems of Cantor set theory, e.g. in the Zermelo-Fraenkel set theory. Some models are easily obtained from ω-saturated models of cardinality $\aleph_1$ of Peano arithmetic. This proves, amongother things, the relative consistency of our theory w.r.t. Zermelo-Fraenkel set theory. Moreover, this enables us to investigate our theory by classical means, to take inspiration from results of model theory and to adopt various proof techniques.

When our theory is understood as a formal system, some of its models can be identified with a rather particular case of Robinson's non-standard analysis, which gives further important sources of proof techniques.

Finally, our theory can be used for a rather particular case of the theory of semisets. This fact also gives us various notions and techniques.

All of these sources of inspiration have been substantionally utilized.

One can object that our theory as a formal system is only a restriction of contemporary mathematics. But this objection is as irrelevant as the objection that the general theory of relativity restricts itself only to one geometry from Klein's classification of geometries.

A formal treatment of our theory can simplify our work in various aspects. But the reader should keep in mind that our theory is <u>not</u> a formal system and that axioms presented in this book do not form an exhaustive list. Only those axioms really used are presented. For example, in Section 6 of Chapter I we present a principle (not exactly formulated) for the acceptance of new axioms. Axioms accepted on the base of this principle make construction of models of our theory in usual set-theoretical systems rather difficult, if not impossible.

4. Mathematics in the alternative set theory

If our theory is to be really an alternative to Cantor set theory we must show that it can replace Cantor set theory in its position in mathematics. Even though the main part of the present book is devoted to the development of the theory itself, the principal aim of our work is to develop mathematics inside it.

We have already mentioned the fact that our theory is a general theory of infinity. After having read the first two chapters the reader will see that objects of pre-set-mathematics can be constructed just as well in our theory as in Cantor set theory. Some doubts might concern only infinitesimal calculus.

First, it is immediately clear that one could develop the ε, δ-analysis in our theory; but functional analysis would be already rather unnatural. However, it is not our intention to develop infinitesimal calculus as the ε, δ-analysis. In our theory, we have for this purpose more natural means, and also functional analysis can be reconstructed in a form more natural for our theory. At present more variants of infinitesimal calculus are being elaborated and it is not yet clear which is the most natural one. This is why infinitesimal calculus has not been included in this book.

Parts of mathematics that were developed after the origin of set theory must be subjected to critical analysis concerning their very foundations. It is necessary to look for new mathematical definitions of notions that have already been mathematically formalized. The chapter on topological shapes is an example of this. Our present experience is that a mechanical adaptation of notions used in Cantor set theory often retards the development and hides simple natural approaches. Here is the main difference between the work in our theory and in Robinson's non-standard analysis. Non-standard analysis is not an informal theory and it derives its raison d'etre from Cantor set theory. Its aim is to enrich Cantor set theory by new techniques. Notions of Cantor set theory are not subjected to any criticism.

Our theory makes possible a natural mathematical treatment of notions that either have not yet been defined mathematically or that have been defined in an unsatisfactory way. As an example we have here the chapter dealing with motion. We shall see that the development of mathematics in the alternative set theory quickly leads to problems for which contemporary mathematics presents no appropriate means for solution. Thus it will be necessary to look for new, unusual techniques.

Having obtained some knowledge of the alternative set theory, the reader will discover various possibilities of development of particular mathematical disciplines in it. Several such possibilities have been investigated in the Prague seminar of the alternative set theory, and they were subjected to severe criticism. This book contains only some examples elaborated by the present author.

Restriction of mathematical problems and concentration on problems posed by the alternative theory is drastic. The spirit of the alternative set theory seems to regulate mathematical problems "in medias res". To solve such problems is by no means easy. If our way or a similar way were shown to be right, then this fact would probably lead to a considerable restriction of mathematical production.

Chapter I

Introduction to Alternative Set Theory

The mathematician creates objects and interrelates them. He does this in various ways, which we shall not attempt to describe in full generality.

The objects and relations that form the subject of mathematical study exist in our minds. For various purposes mathematicians create complex worlds of such objects. Our study will be devoted to a particular world of mathematical objects.

We shall describe and investigate our world of objects informally. But we shall try to choose formulations admitting easy formalization since some formalization will be desirable in later stages of the development of our theory. (By a formalization we mean a formal axiomatic system corresponding to the informal theory in question or at least to some important fragments of that theory.) In particular, properties given to our world in the course of its creation will be spelled out as axioms. Other properties will then be proved from the axioms. Some axioms will be formulated in a way which does not admit a mechanical literal formalization; nevertheless, they will serve as an important orientation.

Objects will be denoted by letters, possibly indexed, or by other standard symbols. $X = Y$ means that X and Y denote the same object and is read "X equals Y". $X \neq Y$ means that X and Y denote distinct objects.

In order to simplify notation we shall use logical connectives $\&$, $\vee$, $\Rightarrow$, $\equiv$, $\neg$ and quantifiers $\exists$, $\forall$, $\exists!$. The quantifier $\exists!$ is read "there is exactly one". Furthermore, we shall use the notation $\varphi(X)$ and $\varphi(X_1, \ldots, X_n)$ to denote properties of objects and relations among objects, respectively. (X and $X_1, \ldots, X_n$ are variables varying over objects.)

The membership relation will play a fundamental rôle in our theory. This relation as well as the objects of our world will be introduced later, but we make already now the convention that the symbol $\in$ denotes the membership. We read $X \in Y$ as "X is a member of Y" or, equivalently, "X belongs to Y", or "X is an element of Y". We write $X \notin Y$ if X does not belong to Y. We shall occasionally use bounded quantifiers $(\exists X \in Y)(\ldots)$ as an abbreviation for $(\exists X)(X \in Y \& \ldots)$ and $(\forall X \in Y)(\ldots)$ as an abbreviation for $(\forall X)(X \in Y \Rightarrow \ldots)$.

Section 1

Sets

Sets are specific objects. We describe their construction and the construction of the membership relation between objects and sets.

First, we assume that the empty set (having no elements) has been constructed. Thus there is an object that is a set and has no elements. This set will be denoted by $\emptyset$, as usual.

Other sets will be constructed as follows. Assume that some objects have been constructed earlier and that we can arrange these objects into a list or, at least, such a list can be imagined. Under these conditions, a new set is said to have been constructed, namely the set of all objects from the list; this set is distinct from each object from the list. The new set does not depend on the order of the elements in the list but is uniquely determined by its elements. Thus if X and Y are sets and have the same members, then X = Y.

Thus our concept of set is similar to Cantor's, but all our sets are finite from Cantor's point of view. We shall not admit the fiction of actually infinite sets.

If $X_1, \ldots, X_n$ is a list of objects then $\{X_1, \ldots, X_n\}$ denotes the set containing exactly all objects $X_1, \ldots, X_n$.

The universe of sets is formed by sets constructed iteratively from the empty set.

We do not claim that the universe of sets contains all possibly constructible sets. Our restriction to the universe of sets in the above sense is not substantial for the purposes of set theory. We have in this universe enough sets to code various objects not belonging to it and in this way reduce problems concerning more general sets (i.e. sets obtainable by our construction from arbitrary objects) to problems concerning the universe of sets. On the other hand, our restriction has several technical advantages.

We shall use lower-case letters to denote sets from the universe of sets. Thus speaking of an object x we automatically assume that x belongs to the universe of sets. The possibility of denoting sets by other symbols is of course not excluded. The fact that X is a set is denoted by Set(X).

Now we are going to formulate axioms which will describe some basic properties of the universe of sets. We shall try to deduce other statements (or their negations) from our axioms rather than to decide their truth directly from the basic intuition behind the universe of sets.

Axiom of extensionality for sets

$(\forall x)(\forall y)(x = y \equiv (\forall z)(z \in x \equiv z \in y))$.

For any sets x and y, x equals y iff x and y have the same elements.

Axiom of the empty set

$(\exists x)(\forall Y)(Y \notin x)$.

There is a set having no elements.

By the axiom of extensionality, there is exactly one empty set.

Axiom of set-successors

$(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \equiv (u \in x \vee u = y))$.

For any sets x and y there is a set z having exactly the following elements: y and all elements of x. By the axiom of extensionality, z is uniquely determined by x and y; we shall write $z = x \cup \{y\}$. Note that $\emptyset \cup \{y\} = \{y\}$, $\{x_1\} \cup \{x_2\} = \{x_1, x_2\}$, etc.

These three axioms express facts stated, at least implicitly, in the course of the construction of the universe of sets. Thus if our construction is not senseless the axioms are true. Axioms of this kind are called analytic axioms.

We shall now consider set-theoretical properties and relations, i.e. properties and relations expressible by means of set-formulas. Set-formulas are expressions constructible by means of the following rules:

1/ $x = y$ and $x \in y$ are set-formulas.

2/ If φ and ψ are set-formulas then $(\varphi \,\&\, \psi)$, $(\varphi \vee \psi)$,

$(\varphi \Rightarrow \psi)$, $(\varphi \equiv \psi)$, $\neg(\varphi)$, $(\exists x)\varphi$, $(\forall x)\varphi$ are set-formulas.

3/ If x and y are replaced in 1/, 2/ by other lower-case variables then the resulting formulas are also set-formulas.

We shall now formulate the last axiom concerning the universe of sets, the axiom of induction. By accepting this axiom we subject our sets to the laws valid in Cantorian set theory for finite sets.

Axiom of induction

Let $\varphi(x)$ be a set-theoretical property. Then

$$\varphi(\emptyset) \,\&\, (\forall x)(\forall y)(\varphi(x) \Rightarrow \varphi(x \cup \{y\})) \Rightarrow (\forall x)\varphi(x)$$

(If φ holds for $\emptyset$ and whenever φ holds for x then φ holds for $x\cup\{y\}$, then φ holds for all x.)

Here we do not have a single axiom but an axiom schema. By substituting various particular set-formulas for φ we obtain particular axioms.

The axiom of induction is not an analytic axiom since by accepting it for sets we ascribe to them various properties that cannot be directly perceived from the definition. Such properties agree with the classical theory of finite sets but they are problematical. The axiom of induction is a postulated hypothesis about finite sets. Axioms of this type will be called hypothetical axioms.

Even if we did not assume the axiom of induction, we could reach the same goals; but various parts of the theory, that can be easily handled using traditional techniques, would become technically quite complicated. Thus we assume the axiom of induction mainly for the sake of convenience. We shall show later how to get rid of it.

For the present, we shall not assume any other axioms for sets. At the end of this section we shall assume one more axiom, the axiom of regularity, which manifests the fact the empty set is the basis for the construction of the universe of sets. But in what follows we shall not need the axiom of regularity.

Some properties of sets and relations among sets are rather frequently used and therefore deserve special notations and names.

For example, we define

$$x \subseteq y \equiv (\forall z)(z \in x \Rightarrow z \in y)$$

(x is a <u>subset</u> of y iff each element of x is an element of y);

$$x \subset y = x \subseteq y \,\&\, x \neq y$$

(x is a <u>proper subset</u> of y iff it is a subset of y distinct from y).

In this book we shall omit trivial and routine theorems about sets; knowledge of them will be assumed. For other theorems, whose proofs are well known, we shall either omit the proof or give only some hints.

<u>Theorem</u>. For any sets x and x_1, there is a set z that has the following elements: all elements of x, all elements of x_1 and no other elements.

<u>Proof</u>. Our theorem can be expressed formally as follows:

$$(\forall x)(\forall x_1)(\exists z)(\forall u)(u \in z \equiv u \in x \vee u \in x_1).$$

Take an arbitrary x_1. Let $\varphi(x)$ be the set-formula $(\exists z)(\forall u)(u \in z \equiv u \in x \vee u \in x_1)$. First, $\varphi(\emptyset)$ is evident. Assume $\varphi(x)$ and let y be arbitrary. Let z satisfy $(\forall u)(u \in z \equiv u \in x \vee u \in x_1)$. Then we have $(\forall u)(u \in z \cup \{y\} \equiv u \in x \cup \{y\} \vee u \in x_1)$, thus $\varphi(x \cup \{y\})$. By the axiom of induction we have $(\forall x)\,\varphi(x)$, which concludes the proof.

By the axiom of extensionality, the z of the preceding theorem is uniquely determined by x and x_1. Hence we can define the operation of union of two sets as follows:

$$z = x \cup x_1 \equiv (\forall u)(u \in z \equiv u \in x \vee u \in x_1).$$

Note that this includes the operation $x \cup \{y\}$ defined above as a particular case.

<u>Theorem</u>. For each x there is a z whose elements are exactly all elements of elements of x.

<u>Proof</u>. Formally, we have to prove:

$$(\forall x)(\exists z)(\forall u)(u \in z \equiv (\exists v \in x)(u \in v)).$$

Let $\varphi(x)$ be $(\exists z)(\forall u)(u \in z \equiv (\exists v \in x)(u \in v))$. Assume $\varphi(x)$ and let y be arbitrary. Let z satisfy $(\forall u)(u \in z \equiv (\exists v \in x)(u \in v))$. Then $(\forall u)(u \in z \cup y \equiv (\exists v \in x \cup \{y\})(u \in v))$, which implies $\varphi(x \cup \{y\})$.

By extensionality, z is determined uniquely by x; hence we can define the union of a set:

$z = \bigcup x \equiv (\forall u)(u \in z \equiv (\exists v)(u \in v \,\&\, v \in x))$.

<u>Theorem</u>. Let $\varphi(u,v)$ be a set-formula and assume $(\forall u)(\exists! v)\varphi(u,v)$. Then $(\forall x)(\exists z)(\forall v)(v \in z \equiv (\exists u \in x)\varphi(u,v))$. In words: Assume that for each u there is exactly one v such that $\varphi(u,v)$; call this v the φ-image of u. Then for each x there is a z whose elements are exactly the φ-images of elements of x.

Strictly speaking, this is not a single theorem but a theorem-schema, called the <u>replacement schema</u>. Substituting various set formulas we obtain particular theorems. Our proof will consist in giving instructions for proving each particular case.

<u>Proof</u>. Let $\psi(x)$ be the set-formula $(\exists z)(\forall v)(v \in z \equiv (\exists u \in x)\varphi(u,v))$. First, $\psi(\emptyset)$ is evident. Assume $\psi(x)$ and take a y. Let z satisfy $(\forall v)(v \in z \equiv (\exists u \in x)\varphi(u,v))$. Let $\varphi(y,y_1)$. Then $(\forall v)(v \in z \cup \{y_1\} \equiv (\exists u \in x \cup \{y\})\varphi(u,v))$, which implies $\psi(x \cup \{y\})$. By the axiom of induction we have $(\forall x)\psi(x)$.

<u>Theorem</u>. $(\forall x)(\exists z)(\forall u)(u \in z \equiv u \subseteq x)$.
In words: For each x, all subsets of x form a set z.

<u>Proof</u>. Let $\varphi(x)$ be $(\exists z)(\forall u)(u \in z \equiv u \subseteq x)$. Assume $\varphi(x)$ and take a y. Let z satisfy $(\forall u)(u \in z \equiv u \subseteq x)$. Note that $(\forall u)(\exists! v)(v = u \cup \{y\})$. Consequently, the replacement principle gives a set z_1 such that $(\forall v)(v \in z_1 \equiv (\exists u)(u \in z \,\&\, v = u \cup \{y\}))$. Then $(\forall u)(u \in z \cup z_1 \equiv u \subseteq x \cup \{y\})$, which implies $\varphi(x \cup \{y\})$.

This theorem enables us to define the power-set operation:
$z = P(x) \equiv (\forall u)(u \in z \equiv u \subseteq x)$.

The following is a theorem-schema.

<u>Theorem.</u> (Comprehension schema.) Let $\varphi(u)$ be a set-formula. Then $(\forall x)(\exists z)(\forall u)(u \in z \equiv u \in x \,\&\, \varphi(u))$. In words: For each x, the elements of x satisfying φ form a set z.

<u>Proof.</u> Let $\psi(x)$ be $(\exists z)(\forall u)(u \in z \equiv u \in x \,\&\, \varphi(u))$. Assume $\psi(x)$ and let y be arbitrary. Let z satisfy $(\forall u)(u \in z \equiv u \in x \,\&\, \varphi(u))$. If $\varphi(y)$ then we set $z_1 = z \cup \{y\}$; otherwise we set $z_1 = z$. Evidently, we have $(\forall u)(u \in z_1 \equiv u \in x \cup \{y\} \,\&\, \varphi(u))$, which implies $\psi(x \cup \{y\})$, which concludes the proof.

Given $\varphi(u)$, the extensionality axiom guarantees that, for each x, the z of the theorem is unique. Thus we can make the following schema of definitions:

For each set-formula $\varphi(u)$, we set
$z = \{u \in x;\ \varphi(u)\} \equiv (\forall v)(v \in z \equiv v \in x \,\&\, \varphi(v))$.

Some particular cases deserve specific notations and names; for example, we set

$x \cap y = \{u \in x;\ u \in y\}$,
$x - y = \{u \in x;\ u \notin y\}$.

Sets x and y are called <u>disjoint</u> if $x \cap y = \emptyset$.

We can now prove with ease that in the universe of sets there is no set containing all sets from that universe.

<u>Theorem.</u> $\neg(\exists x)(\forall y)(y \in x)$, i.e. no set contains all sets as elements.

<u>Proof.</u> Assume $(\forall y)(y \in x)$ and set $v = \{u \in x \,\&\, u \notin u\}$. Then both $v \in v$ and $v \notin v$, a contradiction.

Since $\{X,Y\}$ is the same set as $\{Y,X\}$ for arbitrary objects X and Y, we can call $\{X,Y\}$ the <u>unordered pair</u> of objects X and Y. The <u>ordered pair</u> of X and Y is an object which is determined uniquely by X and Y in this order, and which conversely, determines uniquely X as its first component and Y as its second. This can be achieved, e.g., by setting

$\langle X,Y\rangle = \{\{X\}, \{X,Y\}\}$.

In particular, in the universe of sets we define

$\langle x,y\rangle = \{\{x\}, \{x,y\}\}$.

The following theorem shows that this is a sound definition of the ordered pair:

<u>Theorem.</u> $\langle x,y\rangle = \langle x_1,y_1\rangle \equiv (x = x_1) \,\&\, (y = y_1)$.

The <u>ordered triple</u> is then defined as follows:

$\langle x,y,z\rangle = \langle x, \langle y,z\rangle\rangle$.

Ordered quadruples, quintuples etc. are defined similarly. Now we are ready to define the <u>cartesian product</u> of two sets.

$$x \times y = \{u \in P(P(x \cup y)) ; \ (\exists v_1 \in x)(\exists v_2 \in y)(u = \langle v_1, v_2 \rangle)\}.$$

Note that if $v_1 \in x$ and $v_2 \in y$ then $\{v_1, v_2\} \in P(x \cup y)$ and thus $\langle v_1, v_2 \rangle \in P(P(x \cup y))$. Thus we have

$$u \in x \times y \equiv (\exists v_1 \in x)(\exists v_2 \in y)(u = \langle v_1, v_2 \rangle).$$

We write x^2 instead of $x \times x$.

Having introduced ordered pairs as particular sets we can define the set-theoretical notions of relation and function. We make the following definitions:

$\mathrm{Rel}(x) \equiv (\forall z \in x)$ (z is an ordered pair)
(x is a relation),

$\mathrm{Fnc}(x) \equiv \mathrm{Rel}(x) \,\&\, (\forall u,v,w)\,((\langle v,u \rangle \in x \,\&\, \langle w,u \rangle \in x \Rightarrow v = w)$
(x is a function).

Domain, range etc. can be introduced for arbitrary sets, not just for relations. We make the following definitions:

$\mathrm{dom}(x) = \{u \in \bigcup\bigcup x ; \ (\exists v)(\langle v,u \rangle \in x)\}$,

$\mathrm{rng}(x) = \{u \in \bigcup\bigcup x ; \ (\exists v)(\langle u,v \rangle \in x)\}$

(domain and range).

Observing that $\langle u,v \rangle \in x$ implies $u \in \bigcup\bigcup x$ and $v \in \bigcup\bigcup x$ we obtain immediately the following:

(1) $\mathrm{dom}(x)$ is the set of all u such that $\langle v,u \rangle \in x$ for some v,

(2) $\mathrm{rng}(x)$ is the set of all u such that $\langle u,v \rangle \in x$ for some v.

If f is a function and $u \in \mathrm{dom}(f)$ then $f(u)$ denotes the unique v such that $\langle v,u \rangle \in f$, i.e. $v = f(u)$ iff $\langle v,u \rangle \in f$.

$$x^{-1} = \{u \in P(P(\bigcup\bigcup x)) ; \ (\exists v_1, v_2)(u = \langle v_1, v_2 \rangle \,\&\, \langle v_2, v_1 \rangle \in x)\}$$

(the inverse relation)

$$x \upharpoonright y = x \cap (\mathrm{rng}(x) \times y)$$

(the restriction of x to y).

Obviously, x^{-1} is the set of all pairs $\langle v_1, v_2 \rangle$ such that $\langle v_2, v_1 \rangle \in x$; and $x \upharpoonright y$ is the set of all pairs $\langle v_1, v_2 \rangle \in x$ such that $v_2 \in y$.

Using the set-theoretical notion of a function we can define equivalence (equipollence) and subvalence between sets. We shall speak of set-theoretical equivalence and subvalence or, briefly, of set-equivalence and set-subvalence. The general notions equivalence and subvalence will be introduced in Section 2. We make the following definitions:

A function f is one-one if f^{-1} is also a function.

$x \hat{\approx} y$ (x is set-equivalent to y) iff there is a one-one function f whose domain is x and whose range is y.

$x \hat{\precsim} y$ (x is set-subvalent to y) iff there is one-one function f whose domain is x and whose range is a subset of y.

$x \hat{\prec} y$ (x is strictly set-subvalent to y) iff $x \hat{\precsim} y$ but not $x \hat{\approx} y$.

These notions are commonly known and there are various well-known theorems concerning them. We restrict ourselves to the following three:

Theorem. If y is a proper subset of x then y is strictly subvalent to x.

Proof. We prove $(\forall x)(\forall u \subset x)(\neg u \hat{\approx} x)$. Let $\varphi(x)$ be the set-formula $\neg(\exists u)(u \subset x \,\&\, u \hat{\approx} x)$. Assume $\varphi(x)$ and let $y \notin x$. Let u be a proper subset of $x \cup \{y\}$ such that $u \hat{\approx} x \cup \{y\}$. Take a one-one mapping f of $x \cup \{y\}$ onto u. First, assume $u \subseteq x$. Then $u - \{f(y)\} \subset x$ and $f \restriction x$ is a one-one mapping of x onto $u - \{f(y)\}$, a contradiction. Thus we have $y \in u$ so let $f(z) = y$. Put $g = (f \restriction (x - \{z\})) \cup \{\langle f(y), z \rangle\}$. Then $g \restriction x$ is a one-one mapping of x onto $x \cap u$, and $x \cap u$ is a proper subset of x, a contradiction. We have proved $\varphi(x \cup \{y\})$.

Theorem. If $x \hat{\precsim} y$ and $y \hat{\precsim} x$ then $x \hat{\approx} y$.

Theorem. $x \hat{\precsim} y$ or $y \hat{\precsim} x$.

Proof. Let $\varphi(x)$ be the set-formula $(\forall u)(u \hat{\precsim} x \vee x \hat{\precsim} u)$. We prove $(\forall x)\varphi(x)$ by induction. Assume $\varphi(x)$ and let $y \notin x$. Given u, we have $u \hat{\precsim} x$ or $x \hat{\precsim} u$. In the first case we obtain $u \hat{\precsim} x \cup \{y\}$, hence assume $x \hat{\prec} u$. Consequently, there is a $u_1 \subset u$ such that $x \hat{\approx} u_1$. Let $v \in u - u_1$. We have $x \cup \{y\} \hat{\approx} u_1 \cup \{v\}$, thus $x \cup \{y\} \hat{\precsim} u$, since $u_1 \cup \{v\} \subseteq u$.

Theorem. For each non-empty x, there is a $u \in x$ which is a minimal element of x with respect to inclusion, i.e. there is no $v \in x$ which is a proper subset of u.

Proof (by induction). Let $\varphi(x)$ be the set-formula $x \neq \emptyset \Rightarrow (\exists u \in x)(\forall v \in x)(\neg v \subset u)$. Assume $\varphi(x)$ and let $y \notin x$. If $x = \emptyset$ then $x \cup \{y\}$ has only one element, which is evidently minimal. Thus let $x \neq \emptyset$ and let u be a minimal element of x. If $y \subset u$ let

$u_1 = y$; otherwise let $u_1 = u$. Obviously, u_1 is a minimal in $x \cup \{y\}$.

The following theorem is proved analogously.

<u>Theorem</u>. Each non-empty set has an element which is maximal with respect to inclusion.

Now we formulate the axiom of regularity. This will complete our list of axioms concerning only sets, but the axiom of regularity will not be used in the remaining sections of Chapter I. It is a hypothetical axiom-schema.

<u>Axiom of regularity</u>

Let $\varphi(x)$ be a set-formula. If $(\exists x)\varphi(x)$ then $(\exists x)(\varphi(x) \,\&\, (\forall y \in x)\neg\varphi(y))$.

The conclusion in words: there is a set x satisfying φ such that no element of x satisfies φ. Intuitively, x is the first set satisfying φ obtained in course of the construction of the universe of sets.

<u>Theorem</u>. For all sets u, $u \notin u$.

<u>Proof</u>. Let $\varphi(x)$ be the set-formula $x = u$. Evidently, $(\exists x)\,\varphi(x)$. Let x be such that $\varphi(x) \,\&\, (\forall y \in x)\neg\varphi(y)$, then $x = u \,\&(\forall y \in x)(y \neq u)$, i.e. $(\forall y \in u)(y \neq u)$, which gives $u \notin u$.

We can analogously prove that there are no x and y such that $x \in y \,\&\, y \in x$, no x, y and z such that $x \in y$, $y \in z$ and $z \in x$, etc.

The axiom of regularity can be equivalently replaced by the following.

<u>Axiom of $\in$-induction</u>

Let $\varphi(x)$ be a set formula. If $(\forall x)[(\forall y \in x)\varphi(y) \Rightarrow \varphi(x)]$ then $(\forall x)\varphi(x)$.

To close this section, we show the equivalence of the axiom of regularity and the axiom of $\in$-induction.

First, assume the axiom of regularity. Let $\varphi(x)$ be a set-formula and assume $(\forall x)[(\forall y \in x)\varphi(y) \Rightarrow \varphi(x)]$. Let $\psi(x)$ be $\neg\varphi(x)$. Assume that there is an x not satisfying φ, i.e. $(\exists x)\psi(x)$. By regularity, there is an x such that $\psi(x)$ and $(\forall y \in x)\neg\psi(x)$, i.e. $(\forall y \in x)\varphi(x)$, which implies $\varphi(x)$, a contradiction. We have proved $(\forall x)\varphi(x)$.

Conversely, assume the axiom of $\in$-induction. Let $\varphi(x)$ be a set-formula and assume $(\exists x)\varphi(x)$. Furthermore, assume $(\forall x)(\varphi(x) \Rightarrow (\exists y \in x)\varphi(y))$. Let $\psi(x)$ be $\neg\varphi(x)$. Then $(\forall x)[(\forall y \in x)\psi(y) \Rightarrow \psi(x)]$, hence $(\forall x)\psi(x)$, which contradicts to $(\exists x)\varphi(x)$. This concludes the proof.

Section 2

Classes

Each property of objects can be considered as an object. A property of objects understood as an object is said to be a <u>class</u>. Classes are further specific objects of our study. The fact that an object X is a class is denoted by $\mathrm{Cls}(X)$.

If $\varphi(X)$ denotes a property of objects then $\{X; \varphi(X)\}$ denotes this property as a class; thus $\{X; \varphi(X)\}$ denotes a specific object.

For reasons analogous to our reasons for restricting our study of sets to the universe of sets as described above, we restrict now the domain of classes we are going to study.

The <u>extended universe</u> is formed by classes of the form $\{x; \varphi(x)\}$ where $\varphi(x)$ is a property of sets from the universe of sets. This construction of the extended universe enables us to formulate the following axiom.

<u>Axiom of existence of classes</u>

For each property $\varphi(x)$ of sets from the universe of sets, the extended universe contains the class $\{x; \varphi(x)\}$.

The extended universe depends on our possible restriction of properties admitted in the axiom of existence of classes. Such a restriction is necessary e.g. if we formalize our theory. But at present we shall not make any restriction of the extended universe. Let us stress the basic fact that we shall <u>not</u> confine ourselves to set-theoretical properties (i.e. properties described by set--formulas) in the axiom of existence of classes.

The membership relation between objects and classes will be now defined for the particular case of classes from the extended universe.

If X is an object and if Y is a class from the extended universe (thus Y is a property of sets from the universe of sets) then $X \in Y$ iff X belongs to the universe of sets and X has the property Y.

A considerable technical simplification is achieved if sets from the universe of sets are identified with certain classes. A set y can be identified with the property "x is an element of y". In this way the universe of sets becomes a part of the extended universe. Our convention is expressed by the following axiom:

<u>Axioms of sets as particular classes</u>

$(\forall x)\, Cls(x)$

(in words: each set is a class).

Axioms of the universe of sets formulated in the preceding section as well as all results obtained there become a part of the theory of the extended universe.

The first natural question is what does it mean that two classes X and Y from the extended universe are equal. Evidently this means that X and Y are identical properties. Here one could make various fine distinctions by considering also the syntactic form of the expression of a property. We shall not do that since it is not our aim to develop our study in that direction. We decide that classes X and Y are equal objects if and only if they have the same elements.

Since we shall deal mainly with classes from the extended universe we shall use letters X, Y, ... to denote classes from the extended universe unless explicitly stated otherwise. Similarly, we shall say simply "a class" instead of "a class from the extended universe".

Thus our concept of identity of classes can be described by the following axiom:

<u>Axiom of extensionality for classes</u>

$(\forall X,Y)(X = Y \equiv (\forall u)(u \in X \equiv u \in Y))$.

In words: Two classes are equal iff they have the same elements.

Note that all axioms formulated till now in the present section are analytic.

A class that is not a set is a proper class. Note that if X is a proper class then the set $\{X\}$ is not in the universe of sets.

If $\varphi(x_1, \ldots, x_n)$ is a formula (not necessarily a set formula) concerning sets then we write $\{\langle x_1, \ldots, x_n\rangle;\ \varphi(x_1, \ldots, x_n)\}$ instead of $\{x;\ (\exists x_1, \ldots, x_n)(x = \langle x_1,\ldots,x_n\rangle\ \&\ \varphi(x_1,\ldots,x_n))\}$.

If $\varphi(x)$ is a set formula then we say that the class $\{x;\ \varphi(x)\}$ is set-theoretically definable. Since $y = \{x;\ x \in y\}$, each set is set-theoretically definable.

The universal class is the class $V = \{x;\ x = x\}$. Evidently, V is set-theoretically definable and we have $(\forall x)(x \in V)$. If follows that V is a proper class (not a set).

In Section 1 various notions concerning sets were defined. We now generalize these definitions to definitions of notions concerning classes and add some new definitions.

$X \subseteq Y \equiv (\forall u)(u \in X \Rightarrow u \in Y)$	(inclusion)
$X \subset Y \equiv X \subseteq Y\ \&\ X \neq Y$	(proper inclusion)
$X \cap Y = \{u;\ u \in X\ \&\ u \in Y\}$	(intersection)
$X \cup Y = \{u;\ u \in X \vee u \in Y\}$	(union)
$X - Y = \{u;\ u \in X\ \&\ u \notin Y\}$	(difference)
$X \times Y = \{\langle u,v\rangle;\ u \in X\ \&\ v \in Y\}$	(Cartesian product)
$X^2 = X \times X$	(Cartesian square)
$\mathrm{dom}(X) = \{u;\ (\exists v)\langle v,u\rangle \in X\}$	(domain)
$\mathrm{rng}(X) = \{u;\ (\exists v)\langle u,v\rangle \in X\}$	(range)
$X^{-1} = \{\langle u,v\rangle;\ \langle v,u\rangle \in X\}$	(converse)
$\bigcup X = \{u;\ (\exists v \in X)(u \in v)\}$	(union)
$\bigcap X = \{u;\ (\forall v \in X)(u \in v)\}$	(intersection)
$P(X) = \{u;\ u \subseteq X\}$	(power)
$X \restriction Y = X \cap (V \times Y)$	(restriction)
$X''Y = \{u;\ (\exists v \in Y)(\langle u,v\rangle \in X\}$	(image)
$\mathrm{Rel}(X) \equiv X \subseteq V^2$	(relation)
$\mathrm{Fnc}(X) \equiv \mathrm{Rel}(X)\ \&\ (\forall u,v,w)(\langle v,u\rangle \in X\ \&\ \langle w,u\rangle \in X \Rightarrow v = w)$	(function or mapping)

If F is a function and $x \in \mathrm{dom}(F)$ then $F(x)$ denotes the unique y such that $\langle y,x\rangle \in F$ (the image of x with respect to the function F).

If $\varphi(X)$ is a formula concerning classes then $\bigcap\{X;\ \varphi(X)\}$ denotes the class $\{u;\ (\forall X)(\varphi(X) \Rightarrow u \in X)\}$ and similarly $\bigcup\{X;\ \varphi(X)\}$ denotes the class $\{u;\ (\exists X)(\varphi(X) \,\&\, u \in X)\}$. We shall often use also other expressions, e.g. $\{x \cap y;\ \varphi(x,y)\}$ denotes the class $\{z;\ (\exists x,y)(\varphi(x,y) \,\&\, z = x \cap y)\}$ etc.

If F is a function, $\mathrm{dom}(F) = X$, $\mathrm{rng}(F) = Y$ and F^{-1} is also a function we call F a <u>one-one mapping</u> of X onto Y. We further define:

X is <u>equivalent</u> to Y (notation: $X \approx Y$) if there is a one-one mapping of X onto Y; X is <u>subvalent</u> to Y (notation: $X \precsim Y$) if there is a one-one mapping of X onto a subclass of Y. X is <u>strictly subvalent</u> to Y if $X \precsim Y$ but not $X \approx Y$.

We have obviously $(\forall x)(\forall y)(x \mathrel{\hat{\approx}} y \Rightarrow x \approx y)$ and $(\forall x)(\forall y)(x \mathrel{\hat{\precsim}} y \Rightarrow x \precsim y)$. We shall see later that these implications cannot be reversed and that the following <u>cannot</u> be proved: $(\forall x,y)(x \mathrel{\hat{<}} y \Rightarrow x < y)$.

If R_1 and R_2 are relations and F is a mapping then F is called an <u>isomorphism</u> of $\langle A_1, R_1\rangle$ onto $\langle A_2, R_2\rangle$ if F is a one-one mapping of A_1 onto A_2 and the following holds:

$$(\forall x,y \in A_1)(\langle x,y\rangle \in R_1 \equiv \langle F(x), F(y)\rangle \in R_2).$$

We call $\langle A_1,R_1\rangle$ and $\langle A_2,R_2\rangle$ <u>isomorphic</u> if there is an isomorphism of $\langle A_1,R_1\rangle$ onto $\langle A_2,R_2\rangle$.

A relation R is an <u>ordering</u> of A (in other words, A is <u>ordered</u> by R) iff R is reflexive, antisymetric and transitive on A, i.e.

$$(\forall x \in A)(\langle x,x\rangle \in R),$$
$$(\forall x,y \in A)(\langle x,y\rangle \in R \,\&\, \langle y,x\rangle \in R \Rightarrow x = y),$$
$$(\forall x,y,z \in A)(\langle x,y\rangle \in R \,\&\, \langle y,z\rangle \in R \Rightarrow \langle x,z\rangle \in R).$$

R is a <u>linear ordering</u> of A if, in addition, the following holds:

$$(\forall x,y \in A)(\langle x,y\rangle \in R \vee \langle y,x\rangle \in R).$$

R is a well-ordering of A if R is a linear ordering of A and each non-empty <u>subclass</u> of A has a first element, i.e.

$$(\forall Z)(\emptyset \neq Z \subseteq A \Rightarrow (\exists x \in Z)(\forall y \in Z)(\langle x,y\rangle \in R)).$$

We shall write $We(A,R)$ if A is well-ordered by R.

Let R be a linear ordering of A. A subclass B of A is a <u>segment</u> of A w.r.t. R iff B contains with each element all its predecessors, i.e. $(\forall y \in B)(\forall x \in A)(\langle x\ y \rangle \in R \Rightarrow x \in B)$.

<u>Theorem</u>. If $We(A,R)$ and $B \subseteq A$ then $We(B,R)$.

<u>Theorem</u>. Let $We(A,R)$. A class B is a segment of A w.r.t. R iff either $B = A$ or there is an $x \in A$ such that $B = (R"\{x\} \cap A) - \{x\}$.

<u>Theorem</u>. If $We(A_1,R_1)$ and $We(A_2,R_2)$ and if F, G are isomorphisms of $\langle A_1,R_1 \rangle$ onto $\langle A_2,R_2 \rangle$ then $F = G$.

<u>Proof</u>. Assume $F \neq G$. Let x be the R_1 - first element of A_1 such that $F(x) \neq G(x)$; let $\langle F(x), G(x) \rangle \in R_2$ and let $y \in A_1$ be such that $G(y) = F(x)$. Since $\langle G(y), G(x) \rangle \in R_2$, we have $\langle y,x \rangle \in R_1$, which implies $F(y) = G(y)$ hence $F(y) = F(x)$ and $y = x$, which is a contradiction.

<u>Theorem</u>. If $We(A,R)$ and B is a segment of A w.r.t. R such that $A \neq B$, then $\langle A,R \rangle$ and $\langle B,R \rangle$ are not isomorphic.

<u>Proof.</u> Assume that F is an isomorphism of $\langle A,R \rangle$ onto $\langle B,R \rangle$. If $x \in A - B$ then $x \neq F(x)$ and $\langle F(x), x \rangle \in R$. Let x be the R-first element of A such that $x \neq F(x)$ and $\langle F(x), x \rangle \in R$. Then $F(x)$ is R-less than x, $F(x) \neq F(F(x))$ and $\langle F(F(x)), F(x) \rangle \in R$, which is a contradiction.

<u>Theorem.</u> If $We(A_1,R_1)$ and $We(A_2,R_2)$ then there is an R_1-segment B_1 of A_1 such that $\langle B_1,R_1 \rangle$ and $\langle A_2,R_2 \rangle$ are isomorphic or there is an R_2-segment B_2 of A_2 such that $\langle A_1,R_1 \rangle$ and $\langle B_2,R_2 \rangle$ are isomorphic.

<u>Proof</u>. Let $\varphi(H)$ be a formula saying "$dom(H)$ is an R_1-segment of A_1, $rng(H)$ is an R_2-segment of A_2 and H is an isomorphism of $\langle dom(H), R_1 \rangle$ onto $\langle rng(H), R_2 \rangle$". By preceding theorems, $\varphi(H)$ & $\varphi(H_1)$ implies that $H \subseteq H_1$ or $H_1 \subseteq H$. If $F = \bigcup\{H;\ \varphi(H)\}$, then evidently $\varphi(F)$. It remains to prove $dom(F) = A_1$ or $rng(F) = A_2$. Assume the contrary, let x be R_1-first in $A_1 - dom(F)$ and let y be R_2-first in $A_2 - rng(F)$. Let $F_1 = F \cup \{\langle y,x \rangle\}$, then $\varphi(F_1)$, hence $F_1 \subseteq F$, which implies $x \in dom(F)$, $y \in rng(F)$ - a contradiction.

The following two theorems concern only the universe of sets.

__Theorem__. For each set x there is a linear ordering r of x such that $r \subseteq x^2$.

__Proof__ by induction. For $x = \emptyset$ take $r = \emptyset$. Assume that $r \subseteq x^2$ and r is a linear ordering of x; let $y \notin x$. Then $r' = r \cup [(x \cup \{y\} \times \{y\}]$ is a linear ordering of $x \cup \{y\}$ and $r' \subseteq (x \cup \{y\})^2$.

__Theorem__. If x is linearly ordered by r and z is a non-empty subset of x then z has an r-first and r-last element.

__Proof__. The theorem is trivial for $x = \emptyset$. Assume that the theorem holds for a set x; let $y \notin x$, let r be a linear ordering of $x \cup \{y\}$ and let z be a non-empty subset of $x \cup \{y\}$. If $z = \{y\}$ then y is both the r-first and the r-last element of z; otherwise $z \cap x$ is non-empty and the induction assumption gives a $y_0 \in z \cap x$, r-first in $z \cap x$. If $y \notin z$ or if y is not an r-first element of z then y_0 is the r-first element of z. The existence of a last element is proved analogously.

__Caution__. The preceding theorem does __not__ say that each relation r which is a set and linearly orders a set x is a well-ordering of x. It says only that each non-empty __subset__ of x has a first element; the existence of a non-empty __subclass__ of x which has no first element (and is not a set) is not excluded. See the next section.

Section 3

Semisets

What we have done until now is in no conflict with Cantor's set theory. But in introducing and investigating the notion of a semiset we are leaving the Cantorian set theoretical world.

A semiset is a subclass of a set. We write Sms(X) for "X is a semiset". Thus

$$Sms(X) \equiv (\exists y)(X \subseteq y).$$

Each set is trivially a semiset. A proper semiset is a semiset which is not a set. We are going to show the existence of proper semisets.

Professor Charles Darwin teaches us that there is a set D of objects and a linear ordering of this set such that the first element in that set is an ape Charlie, each non-first element is a son of the immediately preceding element, and the last element is Darwin himself. The collection A of all apes belonging to D is not a set; otherwise A would have a last element. But, as everybody knows, sons of apes are apes; thus every member of D, including Mr. Darwin, would have to be an ape. Elements of D can be coded in the universe of sets, e.g. by $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, ... etc. in such a way that D itself becomes a set from the universe of sets. The class of codes of all apes (element of A) is a proper semiset.

Our example is by no means isolated. Take the property "to be a living man". This is undoubtedly a useful and frequently used property. But we could hardly make a list of all living people. Even if we disregard technical difficulties, such a list is impossible since there is no crisp boundary between not yet born and already born, nor between yet alive and already dead. On the other hand, we can easily imagine a list (set of objects) containing (among others) all living people.

Examples of proper semisets have been known for a long time but they were held for anomalies, as e.g. the "bald man paradox". But we meet proper semisets whenever in considering a property of some objects we emplasize its intension rather than its extension. The reader could himself supply various examples ("to be a table", "to be a book", "to be a beautiful woman" etc.).

Note that also all natural numbers that cannot be described by an English sentence having at most a thousand words form a proper semiset and therefore we cannot claim that this collection has a least element (cf. below).

For various purposes proper semisets can be approximated by sets. This is the subject of approximation theory. Incorrect approximations or disregarding the existence of proper semisets can lead on various levels to questions of the type "which came first, the chicken or the egg".

Thus the existence of proper semisets is a consequence of the axiom of existence of classes. However, when formalizing our theory or applying it to a particular situation we shall restrict the family of properties admitted in the axiom of existence of classes. It could happen that we would eliminate all proper semisets. This is why we guarantee the existence of proper semiset by an axiom.

<u>Axiom of existence of proper semisets</u>

There is a proper semiset. In symbols, $(\exists X)(Sms(X) \& \neg Set(X))$.

<u>Theorem</u>. If X and Y are semisets then the following classes are also semisets: $X \cap Y$, $X \cup Y$, $X - Y$, $X \times Y$, $\bigcup X$, $P(X)$, $dom(X)$, $rng(X)$, X^{-1}, $Y " X$.

<u>Theorem</u>. If X is a set-theoretically definable and is a semiset then X is a set. (Immediate by the results of the preceding section).

In particular, the universal class V is not a semiset.

x - X - x

A mathematical theory that aims to replace Cantor's set theory in its rôle in mathematics must be suitable for the study problems of infinity. Infinity is brought into our theory by means of semisets. But this kind of infinity is different from the actual infinity in Cantor's sense. Our infinity is a phenomenon occuring when we observe large sets. It manifests itself as absence of an easy survey, as our inability to grasp the set in its totality.

One could suspect that proper classes that are not semisets supply an actual infinity, e.g. the universal class V. But this is not the case. Such classes can be understood as merely potentially infinite or, alternatively, can be held for semisets - subclasses of an immense set that has not been included into our universe of sets. Our considerations could be restricted to such a large semiset from the beginning.

Classes that can be perfectly grasped, i.e. have no subclass which is a proper semiset, are finite. Thus we make the following definition: A class X is <u>finite</u> (notation: Fin(X)) iff each subclass of X is a set. Classes that are not finite are called infinite.

Obviously, each finite class is a set. Thus all proper classes, in particular, all proper semisets, are infinite. Note that "X is finite" is not set-theoretical so that it cannot be proved that all sets are finite. The axiom of existence of proper semisets implies that there are infinite sets. (Caution: we stress the fact our notion of infinity differs from the usual Cantorian notion. Recall that all sets are finite in Cantor's sense.)

Since the empty set is its own unique subclass, $\emptyset$ is finite. Similarly each one-element set $\{x\}$ is finite (it has exactly two subclasses, $\emptyset$ and $\{x\}$). If X is finite and Y is a subclass of X then Y is also finite.

<u>Theorem</u>. If x and y are finite then $x \cup y$ is finite.

<u>Proof</u>. Let $X \subseteq x \cup y$. Then $X = (x \cap X) \cup (y \cap X)$. By the assumption both $X \cap x$ and $X \cap y$ are sets. Thus X is also a set.

<u>Theorem</u> (induction for finite sets). Let Z be a class such that $\emptyset \in Z$ and, for each $x \in Z$ and each y, $x \cup \{y\} \in Z$. Then each finite set is in Z.

<u>Proof</u>. Let x be a finite set and let $\leq$ be a linear ordering of x ($\leq$ is a set). Put $z \in Y \equiv (z \in x \ \& \ \{w \in x;\ w \leq z\} \in Z$. We claim $Y = x$. Since $Y \subseteq x$ and x is finite, Y is a set. Assume $Y \neq x$ and let z be the first element of $x - Y$. Then $\{w \in x;\ w < z\} \in Z$ and

$\{w \in x;\ w \leq z\} = \{w \in x;\ w < z\} \cup \{z\}$ belongs also to Z, a contradiction. Thus Y = x and if z_1 is the last element of x we have $x = \{w \in x;\ w \leq z_1\} \in Z$.

Theorem. Let F be a function whose domain is a finite set. Then F is itself a finite set.

Proof. Let Z be the class of all sets x such that each function F whose domain is x is a finite set. We prove using the preceding theorem that Z contains all finite sets. Evidently $\emptyset \in Z$. Suppose $x \in Z$ and $y \notin x$. Let F be a function such that $\mathrm{dom}(F) = x \cup \{y\}$. Then $F = F \upharpoonright x \cup \{\langle F(y), y \rangle\}$. By the assumption, $F \upharpoonright x$ is a finite set; $\{\langle F(y), y \rangle\}$ is obviously a finite set. Thus F is a finite set. This proves $x \cup \{y\} \in Z$.

Theorem. If x is a finite set and $x \approx X$ then X is a finite set and $x \hat{\approx} X$.

Proof. Let F be a one-one mapping of x onto X. By the preceding theorem, F is a set; consequently, $x \hat{\approx} X$ and $X = \mathrm{rng}(F)$ is a set. It remains to prove that X is finite. Let Y be a subclass of X. Put $Z = \{u \in x;\ F(u) \in Y\}$. Since $Z \subseteq x$ and x is finite, Z is a set; since $Y = \mathrm{rng}(F \upharpoonright Z)$ and both F and Z are sets, Y is also a set. Consequently, X is a finite set.

Theorem. If x and y are finite sets then (1) $x \approx y$ iff $x \hat{\approx} y$; (2) $x \preceq y$ iff $x \hat{\preceq} y$; (3) $x \prec y$ iff $x \hat{\prec} y$. The following three theorems are proved using induction for finite sets.

Theorem. If x is finite then P(x) is finite.

Proof. Put $Z = \{x;\ \mathrm{Fin}(P(x))\}$. We have $\emptyset \in Z$. Let $x \in Z$ and $y \notin x$. Put $u = \{v \subseteq x \cup \{y\};\ y \in v\}$. Then $P(x) \hat{\approx} u$, hence u is finite. Since $P(x \cup \{y\}) = P(x) \cup u$, $P(x \cup \{y\})$ is finite. This proves $x \cup \{y\} \in Z$.

Theorem. If x is a finite set of finite sets then $\bigcup x$ is finite.

Proof. Put $Z = \{x;\ (\forall u)(u \in x \Rightarrow \mathrm{Fin}(u)) \Rightarrow \mathrm{Fin}(\bigcup x)\}$. We have $\emptyset \in Z$. Let $x \in Z$ and $y \notin x$. We have $\bigcup(x \cup \{y\}) = (\bigcup x) \cup y$. If each element of $x \cup \{y\}$ is finite then, in particular, each element of x is finite and y is also finite. Thus $\bigcup x$ is finite and $(\bigcup x) \cup y$ is finite. This proves $(x \cup \{y\}) \in Z$.

Theorem. If x and y are finite then $x \times y$ is finite.

Proof. $x \times y \subseteq PP(x \cup y)$.

Theorem. The class of all finite sets is not a semiset.

Proof. Assume $\{x;\ Fin(x)\} \subseteq u$, then $V \subseteq \bigcup u$.

Theorem. A set x is infinite iff for each $y \notin x$ we have $x \approx x \cup \{y\}$.

Proof. If x is finite, $y \notin x$, and $x \approx x \cup \{y\}$ then $x \hat{\approx} x \cup \{y\}$, which contradicts to the results of Section 1. Conversely, let x be infinite and $y \notin x$. Let r be a linear ordering of x. Put $Y = \{u \in x;\ Fin(r''\{u\} \cap x)\}$. For $u \in Y$ let $F(u)$ be the first element of $x - r''\{u\}$ in the ordering r. For $u \in x - Y$ we put $F(u) = u$. Finally, $F(y)$ is the r-first element of x. Evidently, F is a one-one mapping of $x \cup \{y\}$ onto x.

Corollary. If x is infinite and $y \notin x$ then $x \approx x \cup \{y\}$ but not $x \hat{\approx} x \cup \{y\}$; $x \cup \{y\} \preceq x$ but not $x \cup \{y\} \hat{\preceq} x$; and $x \hat{\prec} x \cup \{y\}$ but not $x \prec x \cup \{y\}$.

x - X - x

The axiom of existence of proper semisets does not imply that there are proper semisets included in any specific set. If we guarantee the existence of a proper semiset included in a certain concrete set then we say that we are studying a witnessed universe. When we restrict the family of properties admitted in the axiom of class existence in such a way that proper subsemisets of all concrete sets are eliminated we say that we are studying a limit universe.

The study of witnessed universes is more difficult than that of limit universes. The theory of witnessed universes is in fact inconsistent in the classical sense. If c is an entirely concrete set (say, the set of all natural numbers less than $67^{(293^{159})}$), then it can be obtained in finitely many steps from the empty set by successive addition of single elements; thus c is finite. On the other hand if c has a proper subsemiset then c is infinite in our sense. But our proof of the fact that c is finite has itself infinitely many steps (in our sense); thus it is not convincing. Only proofs of finite length (in our sense) can be said to be convincing. Naturally, in a witnessed universe finite sets cannot be simply isolated. Therefore with each construction in a witnessed universe is associated a degree of convincingness, which decreases as the length and complexity of the construction grows. For example, if we put $d = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}\}$ then the finiteness of the set $P(P(P(d)))$ has a larger degree of convincingness that the finiteness of the same set presented by the list of its elements. Moreover, the proof that these two sets are equal has approximately the same degree of convincingness as the proof that the latter set is finite. The reader will agree that the system of 8 x 8 fields of the chess-board is more comprehensible than the system of the same fields ordered in a linear sequence. We shall not investigate these problems in the present book since they are not yet satisfactorily understood.

Thus we shall study only limit universes. But in the world perceivable by our senses, all situations to which our theory applies correspond to witnessed universes. We shall therefore motivate various notions introduced in our theory by such situations.

Section 4

Countable classes

Our capacity for observation and distinction is limited by the horizon in all directions. Needless to say, this applies not only to optical observation; the horizon is understood in the sense of E. Husserl's "Krisis der europäischen Wissenschaften und die transzendentale Phänomenologie".

If a large set x is observed then the class of all elements of x that lie before the horizon need not be infinite but may converge toward the horizon. The phenomenon of infinity associated with the observation of such a class is called countability.

Thus the following classes are countable: the class of all people we shall meet in our lives, the class of all books we shall have read, the class of all days before our death; also the class of all problems which will be solved by a certain computer, etc.

In our theory, countability is represented by countable classes. The definition is similar to the classical definition in Cantor's set theory, in spite of the different meaning of countability in our theory.

A pair $\langle A, \leq \rangle$ of classes is called an <u>ordering of type ω</u> iff

(1) $\leq$ linearly orders A,

(2) A is infinite, and

(3) for each $x \in A$, the segment $\{y \in A;\ y \leq x\}$ is finite.

<u>Theorem</u>. If $\langle A, \leq \rangle$ is an ordering of the type ω then $\leq$ well-orders A.

<u>Proof</u>. Let Z be a non-empty subclass of A and let $x_0 \in Z$. Since the segment $a = \{y \in A;\ y \leq x_0\}$ is a finite set both $Z \cap a$ and $\leq \cap\, a^2$ are sets. Put $z = Z \cap a$ and $r = \leq \cap\, a^2$. Evidently, $x_0 \in z$ so $z \neq \emptyset$. Since r linearly orders z, z has a first element, say y. We prove that y is a first element of Z in the ordering $\leq$. Let $x \in Z$ and

$x \leq y$. Since $\langle y, x_0 \rangle \in r$ we have $x \leq x_0$, which implies $x \in z$. Thus $\langle x, y \rangle \in r$; since y is an r-first element of z we have $x = y$.

Theorem. If $\langle A, \leq \rangle$ and $\langle A_1, \leq_1 \rangle$ are orderings of type ω, then they are isomorphic.

Proof. By Section 2, since $\leq$ well-orders A and $\leq_1$ well-orders A_1, either the two orderings are isomorphic or a segment of one ordering is isomorphic to the other ordering. Let B_1 be a segment of A_1 (with respect to $\leq_1$) such that $\langle B_1, \leq_1 \rangle$ is isomorphic to $\langle A, \leq \rangle$. Since B_1 is not a finite set, it is not isomorphic to any segment of the form $\{y \in A_1; y \leq x\}$; thus we must have $A_1 = B_1$.

Theorem. Let r be a linear ordering of a (both a and r are sets Then $\langle a, r \rangle$ is not an ordering of type ω.

Proof. Assume that $\langle a, r \rangle$ is an ordering of type ω. Then a is infinite. Let x be the r-last element of a. Then $X = \{y \in a; \langle y, x \rangle \in r\}$ is finite; but $\{y \in a; \langle y, x \rangle \in r\} = a$, a contradiction.

Theorem. Let r be a linear ordering of an infinite set a. Let $X = \{x \in a;$ the segment $\{y \in a; y \leq x\}$ is finite$\}$. Then $\langle X, r \rangle$ is an ordering of type ω and X is a proper semiset.

Proof. Evidently, r linearly orders X and, for each $x \in X$, the segment $\{y \in X; \langle y, x \rangle \in r\}$ is a finite set. Assume that X is a finite set. Then X is a proper subset of a; let y be the r-first element of a - X. Then $X \cup \{y\} = \{z \in a; z \leq y\}$ is a finite set so $y \in X$, a contradiction. Since $X \subseteq a$, X is a semiset. By the preceding theorem, X is not a set.

A class is called countable (notation: Count(X)) iff there is a relation R such that $\langle X, R \rangle$ is an ordering of the type ω. A class is uncountable iff it is neither countable nor finite.

The following is trivial:

Theorem. (1) If X and Y are countable then $X \approx Y$. (2) If X is countable and $X \approx Y$ then Y is countable.

The following theorem is proved exactly in the same way as in Cantor's set theory.

Theorem. (1) If X and Y are countable then $X \cup Y$ and $X \times Y$ are countable. (2) If X is countable and $Y \precsim X$ then Y is either countable or finite. (3) If X is countable and Y is finite then $Y \prec X$.

x - X - x

People have always tried to go beyond the horizon; this is a typical human aspiration. The aim is not merely to shift the horizon further away but to transcend it in the mind. Mathematics is one of the most important instruments for this; it formulates exact statements which transcend the frame work of perception. We shall incorporate a typical principle of transcending the horizon into our theory in the form of an axiom. Let us make the following convention: letters F, G, possibly indexed, will be used to denote functions; lower case letters f, g, possibly indexed, will denote functions that are sets. (This convention holds in the whole book.)

The prolongation axiom

For each countable function F, there is a set function f such that $F \subseteq f$.

The prolongation axiom says that each countable function has a set prolongation. This axiom can be motivated as follows. Imagine that we find ourselves on a long straight road lined with large stones set at regular distances. The stones are numbered by natural numbers; we are situated at stone number 0 and are looking in the direction of increasing enumeration. The guard stones reach the horizon, that is we cannot distinguish any last stone. The function associating with each stone its number is then countable (in our sense). The prolongation axiom assures us that this function has a set prolongation, that is that beyond the horizon there is a stone S such that the stones between the 0-th stone and S form a set and also that the function enumerating these stones is a set. This is, obviously, a hypothesis that cannot be verified unless we leave our stand or improve substantially our observational ability. Similarly we hypothesize that after our death the behavior of the world will at least for a while be similar to its previous behavior. One could give many other examples. The prolongation axiom is a hypothesis which serves as a base for exact knowledge exceeding evidence.

x - X - x

Theorem. Let X be a countable class. Then there are sets a and $\preceq$ such that $\preceq$ is a linear ordering of a and X consists of all $x \in a$ such that the segment $\{y \in a;\ y \preceq x\}$ is finite.

Proof. Let a_1 be an infinite set and let $\preceq_1$ be a set which is a linear ordering of a_1. Let X_1 be the class of all $x \in a_1$ such that the segment $\{y \in a_1;\ y \preceq_1 x\}$ is finite. Then X_1 is countable. Let F be a one-one mapping of X_1 onto X and let $F \subseteq f$. Since $F = f \upharpoonright X_1$ we may assume $dom(f) = a_1$. Let a_2 be the set of all $x \in a_1$ such that the restriction of f to $\{y;\ y \preceq_1 x\}$ is one-one. Evidently, $X_1 \subseteq a_2$. Put $a = f''a_2$; then $X \subseteq a$. Finally define $x \preceq y \equiv (x,y \in a \,\&\, f^{-1}(x) \preceq_1 f^{-1}(y))$. Then a and $\preceq$ have the desired properties.

The last theorem shows that each countable class can be represented as a certain segment of an infinite set with respect to some linear ordering.

The following is trivial:

Theorem. Each countable class is a proper semiset.

Consequently, no set-theoretical definable class is countable; each infinite set is uncountable; each subset of a countable class is finite. The class V is uncountable.

Theorem. If X is a countable class of finite sets then $\bigcup X$ is countable.

Proof. First, $\bigcup X$ is not finite. Indeed, if we had $\bigcup X = u$ where u is a finite set then X would be also finite since $X \subseteq P(u)$. Let $\preceq$ be an ordering of X of type ω. Put
$Y = \{y;\ (\exists x \in X)(y = x - \bigcup\{z \neq x;\ z \preceq x\})\}$. Y is a countable class of finite sets, $\bigcup X = \bigcup Y$ and any two distinct elements of Y are disjoint. Thus assume that any two distinct elements of X are disjoint. Since X is a semiset, $\bigcup X$ is also a semiset. Let u be a set such that $\bigcup X \subseteq u$ and let $\preceq_0$ be a linear ordering of u (a set). For $x,y \in \bigcup X$ put
$x \preceq_1 y \equiv (\exists x_0 \in X)(\exists y_0 \in X)(x \in x_0 \,\&\, y \in y_0 \,\&\, ((x_0 = y_0 \,\&\, x \preceq_0 y) \vee (x_0 \neq y_0 \,\&\, x_0 \preceq y_0)))$.
It is easy to show that $\langle \bigcup X, \preceq_1 \rangle$ is an ordering of type ω.

Theorem. If X is countable then $P(X)$ is also countable.

Proof. We have $X \precsim P(X)$ and therefore $P(X)$ cannot be finite. Let $\langle X, \preceq \rangle$ be an ordering of type ω. Since each subset of X is finite each subset y of X is a subset of a finite segment of X (i.e. there is an $x \in X$ such that $y \subseteq \{ z \in X;\ z \le x \})$. Thus $P(X)$ can be expressed as the union of a countable class of finite sets; by the preceding theorem, $P(X)$ is countable.

Theorem. Let X be a countable class.
(a) $\bigcup X$ is a set iff there is a $z \subseteq X$ such that $\bigcup X = \bigcup z$.
(b) $\bigcap X$ is a set iff there is an $z \subseteq X$ such that $\bigcap X = \bigcap z$.

Proof. We prove (a); (b) can be proved analogously. The implication $\Leftarrow$ is trivial. We prove $\Rightarrow$. Let $\bigcup X = u$. Let a be an infinite set and let $\le$ be a linear ordering of a such that $\le$ is a set and X consists of all $x \in a$ such that the $\le$-segment of a determined by x is finite. Put $a_1 = \{ x \in a;\ (\forall y \in a)(y \le x \Rightarrow y \subseteq u) \}$. Obviously, $X \subseteq a_1$. If $x \in a_1 - X$ then $u = \bigcup X = \bigcup \{ y \in a_1,\ y \le x \}$. Let x_0 be the $\le$-first element of a_1 such that $u = \bigcup \{ y \in a_1;\ y \le x_0 \}$. Then $x_0 \in X$, and consequently $\{ y \in a_1;\ y \le x_0 \} \subseteq X$, i.e. the set $z = \{ y \in a_1;\ y \le x_0 \}$ has all the desired properties.

Corollary. If X is a countable class such that $\bigcap X = \emptyset$ then there is a subset z of X such that $\bigcap z = \emptyset$.

Theorem. Let X and Y be countable disjoint classes. Then there are disjoint sets u and v such that $X \subseteq u$ and $Y \subseteq v$.

Proof. Define a function F on $X \cup Y$ putting $F(x) = \emptyset$ for $x \in X$ and $F(x) = \{ \emptyset \}$ for $x \in Y$. Let $F \subseteq f$ and put $u = \{ x \in \mathrm{dom}(f);\ f(x) = 0 \}$ and $v = \{ x \in \mathrm{dom}(f);\ f(x) = \{ \emptyset \} \}$.

Theorem. Let X and Y be countable classes such that $\bigcup X \cap \bigcup Y = \emptyset$ Then there are disjoint sets u and v such that $\bigcup X \subseteq u$ and $\bigcup Y \subseteq v$.

Proof. Let a be an infinite set and $\le$ its linear ordering (a set); let Z be the countable class of all $x \in a$ such that $\{ y \in a;\ y \le x \}$ is finite. Let F, G be one-one mappings of Z onto X,Y respectively; let $F \subseteq f$ and $G \subseteq g$. Let $\bar{f}(x) = \bigcup (f'' \{ y \in a;\ y \le x \})$ and $\bar{g}(x) = \bigcup (g'' \{ y \in a;\ y \le x \})$, and put $a_1 = \{ x \in a;\ \bar{f}(x) \cap \bar{g}(x) = \emptyset \}$. Then $Z \subseteq a_1$ and the sets $u = \bigcup (f'' a_1)$ and $v = \bigcup (g'' a_1)$ have the desired properties.

Corollary. Let X, Y be countable classes such that $\bigcup X \subseteq \bigcap Y$. Then there is a set u such that $\bigcup X \subseteq u \subseteq \bigcap Y$.

Definition. Let $\varphi(X)$ be a property of classes from the extended universe. The class $\{ X;\ \varphi(X) \}$ is directed (w.r.t. inclusion) iff for each X, Y such that $\varphi(X)$ and $\varphi(Y)$ there is a $Z \supseteq X \cup Y$ such that $\varphi(Z)$. $\{X; \varphi(X)\}$ is dually directed (w.r.t. inclusion) iff for each X, Y such that $\varphi(X)$ and $\varphi(Y)$ there is a $Z \subseteq X \cap Y$ such that $\varphi(Z)$.

In particular, a class Z is directed w.r.t. $\subseteq$ if $(\forall x \in Z)(\forall y \in Z)(\exists z \in Z)(x \cup y \subseteq z)$ and analogously for dual directedness.

Theorem. Let Z be a set-theoretically definable class. If Z is directed then for each subsemiset X of Z there is a $u \in Z$ which is an upper bound of the elements of X w.r.t. inclusion, i.e. there is a $u \in Z$ such that each $v \in X$ is a subset of u. If Z is dually directed then for each subsemiset X of Z there is a $u \in Z$ which is a lower bound of the elements of X w.r.t. $\subseteq$.

Proof. We prove only the first assertion; the second one can be proved analogously. For $Z = \emptyset$ there is nothing to prove. Suppose $X \neq \emptyset$. Since $\bigcup X$ is a semiset, there is a set u_0 such that $\bigcup X \subseteq u_0$. Put $a = \{x \subseteq u_0; (\exists z \in Z)(x \subseteq z)\}$. Observe that $x \in a$ and $y \in a$ implies $x \cup y \in a$; furthermore, $X \subseteq a$ and consequently $a \neq \emptyset$. Hence a has a maximal element w.r.t. inclusion, i.e. there is a $u_1 \in a$ such that u_1 is not a proper subset of any element of a. Take a $v \in a$; then $u_1 \cup v \in a$ and hence $u_1 \cup v = u_1$. There is a $u \in Z$ such that $u_1 \subseteq u$ and this u has the desired property.

The following theorem is analogous but much more powerful.

Theorem. Let Z be a set-theoretically definable class. Let X be a countable subsemiset of Z. If X is directed then there is a $u \in Z$ which is an upper bound of the elements of X w.r.t. inclusion. If X is dually directed then there is a $u \in Z$ which is a lower bound of the elements of X w.r.t. $\subseteq$.

Proof. We prove only the first assertion. Let $\leq$ be a set which is a linear ordering of a set a such that X consists of all $x \in a$ determining a finite segment of a. Put $a_1 = \{x \in a; (\exists u \in Z)(\forall y \in a)(y \leq x \Rightarrow y \subseteq u)\}$. Obviously, $X \subset a_1$. Choose an $x_0 \in a_1 - X$. Then we have a $u \in Z$ such that each element $y \in a$ less then or equal to x_0 is a subset of u. In particular, each element of X is a subset of u.

This theorem has many important corollaries and we shall often use it. We shall call this theorem the theorem on countable directed semisets.

We give an example of application. Let F be a countable function and let Z be set-theoretically definable. If each $f \subseteq F$ belongs to Z then there is an $f \in Z$ prolonging F, i.e. such that $F \subseteq f$. In particular, if F is one-one then putting $Z = \{f;\ f$ is a one-one function$\}$ we obtain a one-one function f prolonging F.

Recall that we claimed in Section 1 that we could omit the Induction axiom from our theory. Then we could define finite sets in the same way even if we could not prove many theorems about them. We could prove the theorem on induction for finite sets (with some effort). Some kind of the axiom of prolongation then makes it possible to transfer set-theoretical properties of finite sets to some infinite sets. For example, not only all finite but also some infinite sets we have their union, power set etc. Our considerations would be then limited to such sets.

Section 5

Codable classes

We have declared that our basic objects of study are classes from the extended universe. Nevertheless, it is sometimes useful to work with other classes. Classes outside the extended universe can serve as a motivational tool; moreover, using such classes often leads to simplified formulations.

In particular, we are led to classes outside the extended universe if there are natural reasons to consider classes whose elements are classes from the extended universe. This section is devoted to the problem of coding of such classes.

If K and S are classes from the extended universe and if S is a relation (subclass of V^2) then $\langle K, S \rangle$ is a <u>coding pair.</u>

Let $\varphi(X)$ be a property of classes from the extended universe. A coding pair $\langle K, S \rangle$ is said to <u>code the class</u> $\{X; \varphi(X)\}$ if, for each class X from the extended universe, we have
$\varphi(X) \equiv (\exists y \in K)(X = S''\{y\})$.

A class $\{X; \varphi(X)\}$ is <u>codable</u> if there is a coding pair which codes that class.

Coding pairs $\langle K, S \rangle$ and $\langle K_1, S_1 \rangle$ <u>code the same class</u> if $(\forall X)((\exists x \in K)(X = S''\{x\}) \equiv (\exists y \in K_1)(X = S_1''\{y\}))$.

<u>Theorem</u>. Let $\{X; \varphi(X)\}$ be a codable class. Let $\psi(X)$ be a property of classes from the extended universe and let ψ imply φ. Then the class $\{X; \psi(X)\}$ is codable.

<u>Proof</u>. Let $\langle K, S \rangle$ code $\{X; \varphi(X)\}$ and put $K_1 = \{x \in K; \psi(S''\{x\})\}$. Then $\langle K_1, S \rangle$ codes $\{X; \psi(X)\}$.

<u>Theorem.</u> The class $\{X; X \subseteq V\}$ is not codable.

Proof. Assume that $\langle K,S\rangle$ codes $\{X;\ X\subseteq V\}$. Put $Y=\{x\in K;\ x\notin S''\{x\}\}$. Then $Y\subseteq V$ and for all $x\in K$ we have $Y\neq S''\{x\}$.

Theorem. Each class from the extended universe is codable.

Proof. Put $E=\{\langle x,y\rangle;\ x\in y\}$. For each X, $\langle X,E\rangle$ codes X.

In Section 2 we defined membership of an object in a class only for classes from the extended universe. Now we extend the definition of membership to codable classes.

Let $\varphi(X)$ be a property of classes from the extended universe. Assume that $\{X;\varphi(X)\}$ is codable. Y belongs to $\{X;\ \varphi(X)\}$, or Y is an element of $\{X;\ \varphi(X)\}$ iff Y has the property φ; in symbols, $Y\in\{X;\ \varphi(X)\}\equiv\varphi(Y)$.

The preceding theorem implies that the present definition extends the definition of membership given in Section 2.

We present some important examples of codable classes.

A class R is an equivalence iff R is a reflexive, symmetric and transitive relation, i.e. iff we have the following:

$$(\forall x)(\langle x,x\rangle\in R),$$
$$(\forall x)(\forall y)(\langle x,y\rangle\in R\equiv\langle y,x\rangle\in R),$$
$$(\forall x)(\forall y)(\forall z)(\langle x,y\rangle\in R\ \&\ \langle y,z\rangle\in R\Rightarrow\langle x,z\rangle\in R).$$

A factorization of a class Z modulo R is the class $Z/R=\{X;(\exists y\in Z)(X=Z\cap R''\{y\})\}$.

The class Z/R need not be a class of the extended universe since some classes of mutually equivalent elements need not be sets. But we have the following trivial theorem:

Theorem. For each Z and each equivalence R, Z/R is coded by $\langle Z,\ R\cap Z^2\rangle$.

Observe that for each equivalence R we have $\mathrm{dom}(R)=V$. We shall often define an equivalence only for elements of a class A. In such a case we always assume that for all $x,\ y\notin A$ we have $\langle x,y\rangle\in R$ (and for all $x\in A,\ y\notin A$ we have $\langle x,y\rangle\notin R$).

We shall take the liberty to denote equivalences by $\doteq$, $=_1$, $=_2$ etc. and write $x\doteq y$ instead of $\langle x,y\rangle\in\ \doteq$ as usual.

Let us make the following definitions:

$Z^Y = \{F,\ dom(F) = Y$ and $rng(F) \subseteq Z\}$
(recall that F varies over functions!),
$P_\omega(Z) = \{X;\ X \subseteq Z$ and X is either finite or countable$\}$.

Note that if Y is a finite set then $Z^Y = \{f;\ dom(f) = Y$ and $rng(f) \subseteq Z\}$ and hence Z^Y is a class of the extended universe. But in non-trivial cases the classes Z^Y and $P_\omega(Z)$ are outside the extended universe.

<u>Theorem</u>. For each Z and each countable Y, the class Z^Y is codable.

<u>Proof</u>. Put $S = \{\langle x,f\rangle;\ Y \subseteq dom(f)\ \&\ rng(f \upharpoonright y) \subseteq Z\ \&\ x \in f \upharpoonright Y\}$.

We show that Z^Y is coded by $\langle dom(S), S\rangle$. If $f \in dom(S)$ then $S''\{f\} = f \upharpoonright Y$; but $f \upharpoonright Y$ is certainly an element of Z^Y. On the other hand, let F be a function such that $dom(F) = Y$ and $rng(F) \subseteq Z$. By the prolongation axiom, there is an f such that $F \subseteq f$. Obviously, $f \in dom(S)$ and $F = S''\{f\}$.

<u>Theorem</u>. $P_\omega(Z)$ is codable for each Z.

<u>Proof</u>. Let Y be a countable class and let $\langle K_1, S_1\rangle$ code Z^Y. We may assume $K_1 \neq V$ (cf. the proof the preceding theorem). Let $a \notin K_1$ and put $K = K_1 \cup \{a\}$. Define
$S = \{\langle x,y\rangle;\ y \in K_1\ \&\ x \in rng\ (S_1''\{y\})\}$.

We show that $\langle K, S\rangle$ codes $P_\omega(Z)$. Let $x \in K$. If $x = a$ then $S''\{a\} = \emptyset$. If $x \neq a$ then $S''\{x\} = rng(S_1''\{x\})$. But $S_1''\{x\}$ is an element of Z^Y, thus $S''\{x\}$ is subclass of Z and is either countable or finite. Conversely, let X be a finite or countable subclass of Z. If $X = \emptyset$ then $X = S''\{a\}$; if $X \neq \emptyset$ then there is a $F \in Z^Y$ such that $X = rng(F)$. Now $F = S_1''\{x\}$ for an $x \in K_1$, which implies $X = S''\{x\}$.

Writing $\{Y_x;\ x \in A\}$ we shall always understand that we have a coding pair $\langle A, S\rangle$ such that $Y_x = S''\{x\}$ for each $x \in A$. Then $\{Y_x;\ x \in A\}$ denotes the class $\{Y;\ (\exists x \in A)(Y = S''\{x\})\}$.

A coding pair $\langle K, S\rangle$ is called <u>extensional</u> iff $x \neq y$ implies $S''\{x\} \neq S''\{y\}$ for each $x,y \in K$. A codable class $\{X;\ \varphi(X)\}$ is <u>extensionally codable</u> iff there is an extensional coding pair which codes $\{X;\ \varphi(X)\}$.

Certainly, each class X of the extended universe is extensionally coded by $\langle X,E\rangle$ (where $E = \{\langle x,y\rangle ; x \in y\}$).

Theorem. (1) If extensional coding pairs $\langle K,S\rangle$ and $\langle K_1,S_1\rangle$ code the same class then $K \approx K_1$. (2) If $\langle K,S\rangle$ is an extensional coding pair and if $K \approx K_1$ then there is an S_1 such that $\langle K_1,S_1\rangle$ is an extensional coding pair which codes the same class as $\langle K,S\rangle$.

Proof. (1) For $x \in K$ put $y = F(x)$ iff $y \in K_1$ & $S''\{x\} = S_1''\{y\}$. Obviously, F is a one-one mapping of K onto K_1. (2) Let F be a one-one mapping of K_1 onto K. Put $S_1 = \{\langle x,y\rangle ; y \in K_1 \ \& \ \langle x,F(y)\rangle \in S\}$.

It is seen that only an extensional coding of a codable class is sufficiently good. Extensionally codable classes behave as if they were classes of the extended universe. For example if $\{X; \varphi(X)\}$ is extensionally codable then we may write $Z \precsim \{X; \varphi(X)\}$ to mean that for an extensional coding pair $\langle K,S\rangle$ which codes $\{X; \varphi(X)\}$ we have $Z \precsim K$. Evidently, it is irrelevant which coding of $\{X; \varphi(X)\}$ is taken. If $\{X; \psi(X)\}$ is another extensionally codable class then $\{X; \varphi(X)\} \precsim \{X; \psi(X)\}$ has the obvious sense.

We constructed the extended universe with the conviction that it is a useful instrument for the investigation of any class having the property that each of its elements can be constructed before the class as whole. We believe that, having such a class $\{X; \varphi(X)\}$ (consisting of various objects), to each object X satisfying $\varphi(X)$ can be associated a set from the universe of sets as its code and in this way the whole class can be coded in the extended universe as the class of all associated codes. This leads to various axioms depending on the specific nature of the classes to be coded. We shall now assume only one axiom; but we do not exclude the possibility of assuming other axioms as well.

Axiom of extensional coding
Each codable class is extensionally codable.

Thus whenever we prove that a class is codable, we may work with an extensional coding.

Theorem. If Y is countable then $P_\omega(Y)$ is uncountable.

Proof. Obviously $P_\omega(Y)$ is not finite. Assume it is countable. Then $P_\omega(Y)$ can be coded by a pair $\langle Y,S\rangle$. Put $Z = \{x \in Y; x \notin S''\{x\}\}$. Evidently $Z \subseteq Y$ but there is no $x \in Y$ such that $Z = S''\{x\}$, a contradiction.

A pair $\langle A, \leq \rangle$ is said to be an <u>ordering of type Ω</u> if

(1) $\leq$ well-orders A,

(2) A is uncountable, and

(3) for each $x \in A$, the segment $\{ y \in A;\ y \leq x \}$ is countable or finit

The following theorems are immediate consequences of theorems in Section 2.

<u>Theorem</u>. If $\langle A, \leq \rangle$ and $\langle A_1, \leq_1 \rangle$ are orderings of type Ω then they are isomorphic.

<u>Theorem</u>. Let $\langle A, \leq \rangle$ be an ordering of type Ω, let B be countable, and let $\leq_1$ well-order B. Then there is a proper segment A_1 of A (w.r.t. $\leq$) such that $\langle A_1, \leq \rangle$ and $\langle B, \leq_1 \rangle$ are isomorphic.

<u>Theorem</u>. Assume that $\leq$ well-orders A; assume further that for each countable B and each well-ordering $\leq_1$ of B there is a proper segment A_1 of A such that $\langle A_1, \leq \rangle$ and $\langle B, \leq_1 \rangle$ are isomorphic. Then A is uncountable.

<u>Theorem</u>. Let $\langle A, \leq \rangle$ be an ordering of type Ω and let $B \approx A$. Then there is an ordering $\leq_1$ such that $\langle B, \leq_1 \rangle$ is an ordering of type Ω.

<u>Theorem.</u> There is an ordering of type Ω.

<u>Proof.</u> Let Y be a countable class; then Y^2 is also countable. Thus $P_\omega(Y^2)$ is extensionally coded by a pair $\langle K, S \rangle$. Put $K_1 = \{ x \in K;\ S''\{x\}$ well-orders $Y \}$ and $\langle x,y \rangle \in S_1$ iff $x, y \in K_1$ and $\langle Y, S''\{x\} \rangle$ is isomorphic to $\langle Y, S''\{y\} \rangle$ Evidently, S_1 is an equivalence. Let $\langle A, S_2 \rangle$ be an extensional coding of K_1/S_1. For $x, y \in A$ put $x \leq y$ iff, for each $x_1 \in S_2''\{x\}$ and each $y_1 \in S_2''\{y\}$, $\langle Y, S''\{x_1\} \rangle$ is isomorphic to a (possibly improper segment of Y w.r.t. the ordering $S''\{y_1\}$. Using the preceding theorems, one easily shows that $\langle A, \leq \rangle$ is an ordering of type Ω.

Section 6

Uncountable classes

In classifying various kinds of infinity we accept also in the alternative theory Cantor's principle that each class determines its basic infinity type - cardinality - independently of any structure the class may be endowed with. This principle is formalized in the definition that two classes have the same cardinality iff they are equivalent.

Let us now ask the question which types of infinity can be found in the extended universe. Our theory offers various possibilities of theories of infinity. For example, we could imitate Cantor's whole theory. In this case we would first assume the axiom of choice which guarantees the well-orderability of the universal class and then define cardinal numbers as segments of such a well-ordering not equivalent to any smaller segment. Further axioms would guarantee the existence of as many cardinal numbers as we liked. This would make it possible to model Cantor's whole theory in our theory. Needless to say, such a model need not contain all subclasses of the class of natural numbers, only some "appropriate" ones. The only difference from Cantor's theory would consist in the understanding of infinite cardinals. Cantor's theory recognizes infinite cardinals almost as a part of our world; but in our theory they are only some more or less pathological semisets.

This example of a development of the alternative theory is not the only possible one. We can assume axioms for any other theory of infinity, provided it does not contradict the other axioms. Cantor's theory is just one possibility.

At present, no reasons for the acceptance of a nontrivial theory of infinity are known. All such theories must be speculative in character. Consequently, their results mentioning infinite cardinalities will be vacuous if their speculative background is rejected. To prevent this, we decide to accept a trivial theory of infinite cardinalities. We shall do this by assuming the following axiom:

Axiom of cardinalities

For all classes X, Y if both X and Y are uncountable then they are equivalent.

This axiom says that there are exactly two infinite cardinalities: countability and uncountability.

We have some more profound reasons for assuming this last axiom. Our aim is to construct the extended universe to be as rich as possible. Only then it can play the role of a universal instrument for the study of mathematical structures. Thus we may postulate the existence of classes corresponding to properties we are unable to describe, provided that their existence is not absurd. Such classes are possible, may exist. Of course, they are imaginary in a sense. Saying that the existence of such and such classes is not absurd we mean that the assumption of their existence does not contradict axioms that have already been assumed. We may formulate a principle serving as a guide for the formulation of new axioms. The principle says that each class that may exist does exist. In particular, the axiom of cardinalities asserts that for any pair of uncountable classes X, Y there is a one-one mapping of X onto Y. Consistency of this axiom relative to the previous axioms was proved by A.Sochor.

By our last axiom we have not excluded any possible development of a more sophisticated theory of infinity, if such a theory should be desirable. In that case we would study some finer notions of cardinality using equivalences based on one-one mappings of a certain kind. This concerns any other axiom accepted on the base of the above principle.

Our last axiom has several simple consequences.

Theorem. For each uncountable class X, there is a relation R such that $\langle X, R \rangle$ is an ordering of type Ω.

In particular, the universal class V, each infinite set, and the class $P_{\omega}(Y)$ for a countable class are examplex of classes having an ordering of type Ω.

A class X is a selector for an equivalence R iff we have the following:

(1) $(\forall x,y \in X)(\langle x,y\rangle \in R \Rightarrow x = y)$,

(2) $(\forall x)(\exists y \in X)(\langle x,y\rangle \in R)$.

Thus a selector chooses a representative of each element of V/R.

<u>Theorem</u>. X is a selector of an equivalence R iff $\langle X,R\rangle$ extensionally codes V/R.

<u>Theorem.</u> Each equivalence has a selector.

<u>Proof</u>. Let R be an equivalence and let $\lesssim$ be a well-ordering of V. Put $A = \{x;\ (\forall y < x)(\langle x,y\rangle \notin R)\}$. Then A is a selector.

<u>Theorem.</u> Let R be a relation. There is a function F such that $F \subseteq R$ and $\mathrm{dom}(F) = \mathrm{dom}(R)$.

<u>Proof</u>. Let $\leq$ be a well-ordering of V of type Ω. For $x \in \mathrm{dom}(R)$ let $F(x)$ be the $\leq$-first element of the class $R''\{x\}$.

<u>Theorem</u>. Let R be a relation such that $\mathrm{dom}(R)$ is countable and, for each $x \in \mathrm{dom}(R)$, $R''\{x\}$ is a semiset. Then R is a semiset.

<u>Proof</u>. Put $\bar{R} = \{\langle y,x\rangle;\ x \in \mathrm{dom}(R)\ \&\ (R \restriction \{x\}) \subseteq y\}$. Let $F \subseteq \bar{R}$ be a function such that $\mathrm{dom}(R) = \mathrm{dom}(\bar{R}) = \mathrm{dom}(F)$. Using the prolongation axiom, take a set f such that $F \subseteq f$. Then $\langle y,x\rangle \in R$ implies $\langle y,x\rangle \in F(x)$, hence $\langle y,x\rangle \in f(x)$, which implies $\langle y,x\rangle \in \bigcup(\mathrm{rng}(f))$. Thus $R \subseteq \bigcup(\mathrm{rng}(f))$; R is a semiset.

<u>Corollary</u>. The universal class is not a union of countably many semisets.

<u>Theorem</u>. Let $\langle A, \leq\rangle$ be an ordering of type Ω. Let B be a countable subclass of A. Then there is an $x \in A$ such that $B \subseteq \{y \in A\ \&\ y \leq x\}$.

<u>Proof</u>. We may assume $A = V$. Put $R = \{\langle y,x\rangle;\ x \in B\ \&\ y \leq x\}$. Then R is a semiset by the preceding theorem; $\mathrm{rng}(R)$ is a segment of $\leq$ and is also a semiset. Hence $\mathrm{rng}(R) \neq V$ and there is an x such that each element of $\mathrm{rng}(R)$ is less then x. Thus $B \subseteq \mathrm{rng}(R) \subseteq \{y \in A;\ y \leq x\}$.

<u>Corollary</u>. If $\langle A, \leq\rangle$ is an ordering of type Ω then A is not cofinal with any of its countable subclasses.

Theorem. Let R be a relation such that $\mathrm{dom}(R)$ is countable and, for each $x \in \mathrm{dom}(R)$, $R \upharpoonright \{x\}$ is countable. Then R is countable class.

Proof. Let $\leq$ be an ordering of V of type Ω. For $x \in \mathrm{dom}(R)$, let $F(x)$ be the first y such that, for each z, $\langle z,x \rangle \in R$ implies $\langle z,x \rangle \leq y$. Since $\mathrm{rng}(F)$ is at most countable, there is an x such that $\mathrm{rng}(F) \subseteq \{ y;\ y \leq x \}$. Thus $R \subseteq \{ y;\ y \leq x \}$, which implies that R is countable.

Corollary. The union of countably many countable classes is a countable class.

Chapter II

Some traditional mathematical structures

In the course of the development of mathematics, mathematicians have created various worlds of objects (or various kinds of such objects) that have become permanent parts of mathematics, owing to their far-reaching importance. These will generally be called traditional mathematical structures.

Various mathematical structures originated before set theory or, although they were developed in parallel with set theory, have kept their autonomy. Various other structures came into existence in dependence on set theory, but their meaning is not confined to their use in set theory.

Any mathematical theory which claims to encompass the entire world of mathematics must confront the traditional mathematical structures similarly as the Cantor's set theory did, i.e. it must incorporate them in some way.

In our case, we shall construct traditional structures inside the extended universe and shall not be forced to make any further extension of our universe. This way is entirely usual in the Cantor's set theory.

By saying that we construct a traditional structure in the extended universe, we mean that, mutatis mutandis, we construct a canonical model of that structure inside the extended universe. Then we prove several theorems about the model. These convince us that the model can be identified with the original structure in that the model has all the important properties of the structure in question.

However, traditional structures are constructed in the extended universe not only in order to show that such a construction is possible. They will be also used for studying of the extended universe. We shall see that some problems concerning the extended universe can be reduced to problems concerning some traditional structures; thus using traditional structures we shall be able to understand better the structure of the extended universe.

In the present chapter we shall construct only structures that will be used in the sequel.

Section 1

Natural numbers

Following von Neumann, we shall model the natural numbers in the extended universe in such a way that zero is the empty set and each natural number is the set of all smaller natural numbers. Thus we put $0 = \emptyset$, $1 = \{\emptyset\}$, $2 = \{\emptyset, \{\emptyset\}\}$ etc.; in other words we identify 2 with $\{\emptyset, \{\emptyset\}\}$ etc.

This way is obviously rather advantageous. First, each natural number is a set from the universe of sets. A number x is less than a number y iff $x \in y$. And observe that e.g. the number 5 is a set having five elements, which makes possible to define the number of elements of a set as the unique natural number set-theoretically equivalent with that set.

However, our aim is not only to define particular natural numbers but also the class of all natural numbers. Thus we need a property $\varphi(x)$ that can be identified with the property of being a natural number. When we formulate such a property we are obliged to prove various theorems showing that our choice is been adequate.

We shall construct the natural numbers inside the universe of sets. Thus the whole first part of the present section will concern only the universe of sets; if we shall use classes at all then only set-theoretically definable ones; they could be always replaced by the respective set-theoretical properties. For the sake of simplicity we shall use heavily the regularity axiom, even if it is well known that this axiom is not indispensable for the construction of the natural numbers.

A set x is a natural number if it satisfies the following:

(1) Each element of x is a subset of x, i.e. $(\forall y \in x)(y \subseteq x)$, and

(2) $\in$ is connected on x, i.e. $(\forall y,z \in x)(y \in z \vee y = z \vee z \in y)$.

The class of all natural numbers is denoted by N. We use α, β,

γ, δ (possibly indexed) as variables ranging over natural numbers.

We have some trivial facts.

Theorem. Each element of a natural number is a natural number.

Theorem. If $\alpha, \beta \in N$ then $\alpha \cap \beta \in N$.

Theorem. If $\alpha, \beta \in N$ and α is a proper subset of β then $\alpha \in \beta$.

Proof. Assume $\alpha \subset \beta$. By the regularity axiom, there is a $\gamma \in \beta - \alpha$ such that $\gamma \cap (\beta - \alpha) = \emptyset$. Now $\gamma \in \beta$ implies $\gamma \subseteq \beta$ and hence $\gamma \subseteq \alpha$. Let $\delta \in \alpha$. Then $\delta, \gamma \in \beta$, which implies $\delta \in \gamma$ or $\delta = \gamma$ or $\gamma \in \delta$. But the last two cases imply $\gamma \in \alpha$, which is impossible. Thus $\delta \in \gamma$, which proves $\alpha \subseteq \gamma$. We have shown $\alpha = \gamma$ and consequently $\alpha \in \beta$.

Theorem. N is linearly ordered by the relation $\{\langle \alpha, \beta \rangle; \alpha \in \beta \vee \alpha = \beta\}$.

Proof. Put $R = \{\langle \alpha, \beta \rangle, \alpha \in \beta \vee \alpha = \beta\}$. Evidently, R is reflexive. Assume $\langle \alpha, \beta \rangle \in R$, $\langle \beta, \alpha \rangle \in R$ and $\alpha \neq \beta$. Then $\alpha \in \beta$ and $\beta \in \alpha$, hence $\alpha \in \alpha$, which is impossible by the axiom of regularity. Thus $\langle \alpha, \beta \rangle \in R$ and $\langle \beta, \alpha \rangle \in R$ implies $\alpha = \beta$. We prove transitivity. Let $\langle \alpha, \beta \rangle \in R$ and $\langle \beta, \gamma \rangle \in R$. If $\alpha = \beta$ or $\beta = \gamma$ then trivially $\langle \alpha, \gamma \rangle \in R$. If $\alpha \in \beta$ and $\beta \in \gamma$ then $\alpha \in \gamma$ and consequently $\langle \alpha, \gamma \rangle \in R$. Finally, take arbitrary $\alpha, \beta \in N$. Then $\alpha \cap \beta \in N$ and $\alpha \cap \beta$ is a subset both of α and of β. If $\alpha \cap \beta = \alpha$ then $\alpha \subseteq \beta$ and consequently either $\alpha = \beta$ or $\alpha \in \beta$ (use the preceding theorem). Similarly for $\alpha \cap \beta = \beta$. Thus it remains to consider the case $\alpha \cap \beta \subset \alpha$ and $\alpha \cap \beta \subset \beta$. Then $\alpha \cap \beta \in \alpha$ and $\alpha \cap \beta \in \beta$ by the preceding theorem, thus $\alpha \cap \beta \in \alpha \cap \beta$, which is impossible by the axiom of regularity.

When we speak of the ordering of natural numbers we always mean the ordering from the last theorem, if not stated otherwise, and denote it by $\leq$. Let us stress the fact, that $\leq$ is no well-ordering of N.

Theorem. (1) $\emptyset \in N$. (2) For each $\alpha \in N$, $\alpha \cup \{\alpha\}$ is an element of N. (3) The number 0 is the least element of N in the ordering of natural numbers. (4) For each α, the number $\alpha \cup \{\alpha\}$ is the immediate successor of α. (5) If $\alpha \neq 0$ then there is a β such that

$\alpha = \beta \cup \{\beta\}$.

Proof. (1) - (3) are trivial. (4) : Evidently, $\alpha \in \alpha \cup \{\alpha\}$ and $\alpha \neq \alpha \cup \{\alpha\}$. Assume $\alpha \in \gamma \in \alpha \cup \{\alpha\}$. Then either $\gamma \in \alpha$ or $\gamma = \alpha$. In both cases we obtain $\alpha \in \alpha$, which is a contradiction. (5) . Let $\alpha \neq 0$ and consider the relation $\alpha^2 \cap \{\langle\gamma, \beta\rangle;\ \gamma \in \beta \vee \gamma = \delta\}$ (the restriction of the ordering of natural numbers to α). This is a set and a linear ordering of α ; thus there is a largest element β. We have $\beta \in \alpha$ and $\beta \cup \{\beta\} \notin \alpha$. Since $\alpha \in \beta \cup \{\beta\}$ is impossible, we get $\alpha = \beta \cup \{\beta\}$.

Consequently, N has no largest element in the ordering of natural numbers and hence N is not a set.

The theorems we have proved show that our natural numbers have all the ordinal properties that we expect from natural numbers. In the following theorem we shall show that our natural numbers have also the necessary cardinal properties.

Theorem. For each set x there is a unique natural number α set-theoretically equivalent to x.

Proof. We first show the existence using the axiom of induction. Let $\varphi(x)$ be the set-formula saying $(\exists\alpha)(x \hat{\approx} \alpha)$. Then $\varphi(\emptyset)$ is evident. Assume $\varphi(x)$ and let $y \notin x$. Let α be such that $x \hat{\approx} \alpha$. Then $x \cup \{y\} \hat{\approx} \alpha \cup \{\alpha\}$, which gives $\varphi(x \cup \{y\})$. It remains to prove uniqueness. Assume $x \hat{\approx} \alpha$ and $x \hat{\approx} \beta$. Then $\alpha \hat{\approx} \beta$. Now if $\alpha \in \beta$ then α is a proper subset of β ; thus $\alpha \hat{\approx} \beta$ is impossible. Similarly, $\beta \in \alpha$ is impossible. Consequently, $\alpha = \beta$.

By the last theorem, for each set x we can define the number of elements of x as the unique natural number α such that $x \hat{\approx} \alpha$.

Addition and multiplication of natural numbers are defined as follows:

$\gamma = \alpha + \beta \equiv \gamma \hat{\approx} \alpha \cup (\{\beta\} \times \beta)$,
$\gamma = \alpha \cdot \beta \equiv \gamma \hat{\approx} \alpha \times \beta$, and
$\alpha + 1 = \alpha \cup \{\alpha\}$.

Note that the last definition agrees with the definition of sum and with the definition of 1.

We may define exponentiation by taking α^β to be the number of elements of the set $\{f;\ \mathrm{dom}(f) = \beta \text{ and } \mathrm{rng}(f) \subseteq \alpha\}$.

It is easy to verify that the class N endowed with the operations of successor, sum, and product satisfies the axioms of first order Peano arithmetic. This shows that we have constructed the natural numbers inside the universe of sets in such a way that they can be identified with the classical natural numbers.

Now we are going to prove the theorem on recursive definition of functions on natural numbers; this theorem is an important example of an application of the natural numbers in the universe of sets.

Theorem. Let G be a set-theoretically definable function and let $dom(G) = V$. Then there is exactly one function F such that $dom(F) = N$ and $F(\alpha) = G(F \upharpoonright \alpha)$ for each α; this function is set--theoretically definable.

Proof. Put $M = \{f; dom(f) \in N \ \& \ (\forall \alpha \in dom(f))(f(\alpha) = G(f \upharpoonright \alpha))\}$. Obviously, M is set-theoretically definable. Let f, $g \in M$. Then $f \subseteq g$ or $g \subseteq f$. Indeed, if α were the least number such that $\alpha \in dom(f) \cap dom(g)$ and $f(\alpha) \neq g(\alpha)$ then we would have $f \upharpoonright \alpha = g \upharpoonright \alpha$, thus $f(\alpha) = G(f \upharpoonright \alpha) = G(g \upharpoonright \alpha) = g(\alpha)$, a contradiction. We prove that for each α there is an $f \in M$ such that $\alpha \in dom(f)$. Otherwise, let α be the least number for that such f does not exist. We have $\alpha \neq 0$, thus $\alpha = \beta \cup \{\beta\}$ for some β. Let $f \in M$ and $\beta \in dom(f)$. Then $\beta \cup \{\beta\} \notin dom(f)$, thus $\alpha = dom(f)$. Put $g = f \cup \{\langle G(f), \alpha \rangle\}$; we have $g \in M$ and $\alpha \in dom(g)$. Now put $F = \bigcup M$. We see that F has all desired properties. It remains to prove uniqueness. Let F_1 be such that $dom(F_1) = N$ and $(\forall \alpha)(F_1(\alpha) = G(F_1 \upharpoonright \alpha))$ and assume $F \neq F_1$. Let α be the least number such that $F(\alpha) \neq F_1(\alpha)$. Then $F \upharpoonright \alpha = F_1 \upharpoonright \alpha$, thus $F(\alpha) = G(F \upharpoonright \alpha) = G(F_1 \upharpoonright \alpha) = F_1(\alpha)$, a contradiction. This concludes the proof.

There are several variants of the theorem on recursive definition of functions; these variants can be easily deduced from the last theorem or can be proved in the same way. For example, we may replace the condition $(\forall \alpha)(F(\alpha) = G(F \upharpoonright \alpha))$ by the condition $F(0) = G(0) \ \& \ (\forall \alpha)(F(\alpha + 1) = G(F(\alpha)))$ etc.

Using the last theorem we define the hierarchy of iterated power-sets and associate with each set its rank. These notions help to analyze the structure of the universe of sets.

The following defines set-theoretically a unique function on natural numbers:

$\overline{P}(0) = \{\emptyset\}$, $\overline{P}(\alpha + 1) = P(\overline{P}(\alpha))$.

<u>Theorem</u>. $\alpha \in \beta$ implies $\overline{P}(\alpha) \subseteq \overline{P}(\beta)$.

<u>Proof</u>. We prove $\overline{P}(\alpha) \subseteq \overline{P}(\alpha + 1)$ for each α. Assume the contrary and let α be the least number such that $\overline{P}(\alpha) \nsubseteq \overline{P}(\alpha + 1)$. Since $\alpha \neq 0$, there is a β such that $\alpha = \beta + 1$; we have $\overline{P}(\beta) \subseteq \overline{P}(\beta + 1)$. But this implies $P(\overline{P}(\beta)) \subseteq P(\overline{P}(\beta + 1))$, thus $\overline{P}(\alpha) \subseteq \overline{P}(\alpha + 1)$, a contradiction.

<u>Theorem</u>. $V = \bigcup rng(\overline{P})$.

<u>Proof</u>. Put $V' = \bigcup rng(\overline{P})$. Evidently, $\emptyset \in V'$. Assume $y \subseteq V'$. Let f be a function on y defined as follows: $f(x) = \min\{\alpha;\ x \in \overline{P}(\alpha)\}$. There is a γ such that $f''y \subseteq \gamma$. Hence for each $x \in y$ we have $x \in \overline{P}(\gamma)$, which implies $y \in \overline{P}(\gamma + 1)$. We have $y \in V'$. This proves $V = V'$ by the axiom of $\in$-induction.

This theorem shows that we may associate with each set x from the universe of sets its <u>rank</u> $\tau(x)$, defined as the least natural number α such that $x \in \overline{P}(\alpha)$. The following is obvious:

<u>Theorem</u>. (1) For each natural number α, $\tau(\alpha) = \alpha$.
(2) $x \in y$ implies $\tau(x) \in \tau(y)$.
(3) $\tau(x) = (\max\{\tau(y);\ y \in x\}) + 1$.

<u>Theorem</u>. There is a set-theoretically definable one-one mapping G of $P(N)$ onto N such that $G(\emptyset) = 0$ and $x \subset y$ implies $G(x) \in G(y)$.

<u>Proof</u>. Put $G(x) = \sum\{2^{\alpha};\ \alpha \in x\}$ for each $x \subseteq N$; this mapping has the desired properties.

<u>Theorem</u>. There is a set-theoretically definable one-one mapping of N onto V.

<u>Proof</u>. Let G be the mapping from the preceding theorem. F is defined inductively by the condition $F(\alpha) = F''(G^{-1}(\alpha))$. Using the axiom of $\in$-induction we prove $rng(F) = V$.

- x - X - x -

In the rest of this section, we shall consider natural numbers from the point of view of the extended universe.

<u>Theorem</u>. There is an infinite natural number.

Proof. Let x be an infinite set and let α be the number of elements of x, i.e. $x \hat{\approx} \alpha$. Obviously, α is infinite.

FN will denote the class of all finite natural numbers, i.e. $FN = \{\alpha ; Fin(\alpha)\}$. Evidently, $FN \subset N$.

Letters n, m, k, possibly indexed, will be used as variables for finite natural numbers.

By Chapter I, Section 3, we have immediately the following:
$0 \in FN \;\&\; (\forall n)((n + 1) \in FN)$;
$(\forall m,n)((m + n) \in FN)$; $(\forall m,n)((m.n) \in FN)$;
$(\forall m,n)(m^n \in FN)$; $(\forall n)(n \subseteq FN)$.

Theorem. The ordering of natural numbers $\{\langle \alpha, \beta \rangle ; \alpha \in \beta \vee \alpha = \beta\}$ restricted to FN is an ordering of type ω.

Proof. FN is infinite since it has no last element in the ordering of natural numbers. For each $n \in FN$, the segment $\{\alpha \in N; \alpha \leq n\}$ is $n \cup \{n\}$, hence a finite set.

Consequently, FN is a countable class. Note that N is uncountable since it is a set-theoretically definable proper class.

The following is the theorem on induction for finite natural numbers.

Theorem. For each class X, if $0 \in X$ and if X contains with each n also n + 1, then $FN \subseteq X$.

The easy proof is left to the reader. Now we are going to formulate a theorem on recursive definition for finite natural numbers; we shall present a rather powerful formulation.

Theorem. Let $\varphi(X,Y)$ be a formula such that $(\forall Y)(\exists! X)\varphi(X,Y)$. There is a unique relation R such that $dom(R) \subseteq FN$ and for each n we have $\varphi(R''\{n\} ; R \restriction n)$.

Proof. Let $\psi(X,n)$ be the formula
$Rel(X) \;\&\; dom(X) \subseteq n \;\&\; (\forall m \in n)\; \varphi(X''\{m\} ; X \restriction m)$.
It follows immediately that $\psi(X,n) \;\&\; \psi(Y,n)$ implies X = Y and that $\psi(X,n) \;\&\; m \in n$ implies $\psi(X \restriction m, m)$. We prove that for each n there is an X such that $\psi(X,n)$. Assume the contrary and let n be the least number such that $\neg(\exists X)\psi(X,n)$. (Here we use the fact that each element of FN is finite!) Evidently, $n \neq \emptyset$; let n = m + 1 and

let Y be such that $\psi(Y,m)$. Take the unique Y_0 satisfying $\varphi(Y_0,Y)$ and put $X = Y \cup (Y_0 \times \{m\})$. Then $\psi(X,n)$, a contradiction. Putting $R = \bigcup\{X;\ (\exists n)\ \psi(X,n)\}$, we obtain a relation with the desired properties. Uniqueness is then obvious.

The following theorem is another variant of the theorem on recursive definition; it is proved similarly.

Theorem. Let G be a function such that $\mathrm{dom}(G) = V$. Then there is a unique function F such that $\mathrm{dom}(F) = FN$ and $F(n) = G(F \restriction n)$ for each n.

One can easily verify that not only all natural numbers but also all finite natural numbers satisfy all axioms of first-order Peano arithmetic. Thus we have two models of natural numbers in the extended universe, the elements of N on the one hand, and the elements of FN on the other hand.

Note that only elements of N play the role of natural numbers in the universe of sets. If we work in an extended universe which is a limit universe then the role of the classical natural numbers is played by the elements of FN. In this case we can view N as a useful prolongation of natural numbers, which preserves many good properties of natural numbers but is not well-ordered since e.g. the class N - FN has no least element. Finally, if we work in a witnessed universe then the classical natural numbers correspond to elements of N, whereas FN forms a canonical representative of the way to the horizon. Since we have to motivate our considerations mostly by witnessed universes, we shall prefer the terminology "natural numbers" and "finite natural numbers".

Section 2

Rational and real numbers

Since we have distinguished natural numbers from finite natur numbers, we shall have also two kinds of rational numbers. We shal only sketch their construction since it is quite usual.

First, we construct the integers (positive, negative and zero and the finite integers. For this purpose we take 0 as a code for the sign "minus". Then the class $N^{(-)}$ of all integers and the clas $FN^{(-)}$ of all finite integers are defined as follows:

$$N^{(-)} = N \cup \{\langle 0,\alpha\rangle ;\ \alpha \neq 0\},$$

$$FN^{(-)} = FN \cup \{\langle 0,n\rangle ;\ n \neq 0\}.$$

Obvious definitions of sum and product of (finite) integers and proofs of simple basic theorems on them are left to the reader Note only that $FN^{(-)}$ is a subclass of $N^{(-)}$ and that the operations on $FN^{(-)}$ are restrictions of the corresponding operations on $N^{(-)}$ $FN^{(-)}$ is countable and $N^{(-)}$ is uncountable. $N^{(-)}$ and sum and produ as operations on $N^{(-)}$ are set-theoretically definable.

In speaking about fields, integral domains, etc. we make the convention that by "the field T" we mean a class (domain) endowed with two operations (satisfying the usual axioms) but by "the clas T" we mean only the domain.

The field RN of rational numbers (rationals) is defined as the quotient field of the integral domain $N^{(-)}$. Clearly, we can define the field RN in such a way that it is set-theoretically definable and that $N^{(-)}$ is a substructure of RN.

The field RN has its natural linear ordering, which is also set-theoretically definable; it will be denoted by $\leq$. We introduc positive and negative rationals, absolute value (notation: $|x|$) et All other notions concerning rationals are denoted as usual.

The class FRN of finite rational numbers is defined as follows: $FRN = \{\frac{x}{y} ; x,y \in FN^{(-)} \ \& \ y \neq 0\}$.

Restricting sum and product to FRN we make FRN into the field of finite rational numbers.

Obviously, the field FRN is the quotient field of the integral domain $FN^{(-)}$. The natural ordering of FRN is the restriction of the natural ordering of RN to FRN. The absolute value defined in FRN is the restriction of the absolute value defined in RN, etc. The class FRN is countable and the class RN is uncountable.

In a limit universe, the field FRN corresponds to the classical rationals. On the other hand, in a witnessed universe, the role of classical rationals is played by RN.

The class BRN of bounded rationals (or: finitely large rationals) is defined as follows:
$BRN = \{x; \ x \in RN \ \& \ (\exists n)(|x| < n)\}$.
Obviously, $FRN \subseteq BRN \subseteq RN$.

<u>Theorem.</u> Let $x \in BRN$ and $0 \neq y \in FRN$. Then there is a finite natural number n such that $|x| < n|y|$.

<u>Proof</u>. We may assume $x,y > 0$. There are n_1, n_2, n_3 such that $n_1 n_2 \neq 0$, $x < n_3$ and $y = \frac{n_1}{n_2}$. Let n be such that $nn_1 > n_2 n_3$. Then we have evidently $x < ny$.

<u>Theorem</u>. Let X be a non-empty class of finite rational numbers containing with each finite rational number x all finite rational numbers $y \leq x$. Then there is a rational number z such that $X = \{x \in FRN; \ x \leq z\}$. If $X \neq FRN$ then z is bounded.

<u>Proof</u>. If $X = FRN$ then, for each $\alpha \in N - FN$, $X = \{x \in FRN; \ x \leq \alpha\}$, thus we shall assume $X \neq FRN$. If X has a maximal element in the natural ordering $\leq$ of rationals then let z be this maximal element. If FRN - X has a minimal element z_1 in $\leq$ then put $z = z_1 - \frac{1}{\alpha}$ for an infinite natural number α . Thus assume that FRN - X has no minimum and X has no maximum. For each non-zero n, let F(n) be the largest finite integer such that $\frac{F(n)}{2^n} \in X$. The existence of such a number is an easy consequence of the preceding theorem. Obviously, $\frac{F(n)}{2^n} \leq \frac{F(n+1)}{2^{n+1}} < \frac{F(n)+1}{2^n}$. If $x \in X$ then there is an $n \neq 0$ such that

$x < \frac{F(n)}{2^n}$. This follows from the fact that X has no maximum. Indeed, there is an $y \in X$ larger than x and an $n \neq 0$ such that $\frac{1}{2^n} < y - x$ and hence $x < \frac{F(n)}{2^n}$. Analogously, since FRN - X has no minimum, for each $x \in$ FRN - X there is an $n \neq 0$ such that $\frac{F(n)+1}{2^n} < x$. By the axiom of prolongation, there is a function f such that $F \subseteq f$, dom(f) is an infinite natural number γ, and for each $\alpha < \gamma$ we have $\frac{f(\alpha)}{2^\alpha} \leq \frac{f(\alpha+1)}{2^{\alpha+1}} < \frac{f(\alpha)+1}{2^\alpha}$. Pick an infinite $\gamma_0 < \gamma$ and put $z = \frac{f(\gamma_0)}{2^{\gamma_0}}$. Since $\frac{F(n)}{2^n} \leq z < \frac{F(n)+1}{2^n}$ for each finite n, we have $X = \{x \in \text{FRN};\ x \leq z\}$. Moreover, $z \in$ BRN, since $|z| < |F(1) + 1|$.

Two rational numbers x, y are said to be <u>infinitely near</u> (notation: $x \doteq y$) iff they satisfy one of the following conditions:

(a) $|x - y| < \frac{1}{n}$ for each non-zero $n \in$ FN, or

(b) $n < x$ and $n < y$ for each $n \in$ FN, or

(c) $x < -n$ and $y < -n$ for each $n \in$ FN.

Obviously, $\doteq$ is an equivalence.

<u>Theorem</u>. (1) If $x \in$ BRN, $y \in$ RN and $x \doteq y$ then $y \in$ BRN.

(2) For each pair x,y of bounded rationals, $x \doteq y \equiv (\forall n \neq 0)(|x-y| < \frac{1}{n})$.

(3) For each pair x,y of finite rationals, $x \doteq y$ iff $x = y$.

<u>Proof</u> of (3) : If $x \neq y$ then there is an $n \neq 0$ such that $|x - y| > \frac{1}{n}$.

<u>Theorem</u>. Let x and y be bounded rationals, $x < y$ and x not infinitely near to y. Then there is a finite rational number z such that $x < z < y$.

<u>Proof</u>. We may assume $0 \leq x$. Let $n \neq 0$ be such that $\frac{1}{n} < y - x$. Let m be the least natural number such that $x < \frac{m}{n}$. Then we evidently have $\frac{m}{n} < y$.

<u>Theorem</u>. Let α be an infinite natural number. Then for each $x \in$ RN there is a γ such that $-\alpha^2 < \gamma < \alpha^2$ and $x \doteq \frac{\gamma}{\alpha}$.

<u>Proof</u>. Put $\gamma = \max\{\beta;\ -\alpha^2 < \beta < \alpha^2\ \&\ \frac{\beta}{\alpha} < x\}$. Then $x \doteq \frac{\gamma}{\alpha}$.

<u>Theorem</u>. Let x, x_1, y, $y_1 \in$ BRN and assume $x \doteq x_1$ and $y \doteq y_1$.

Then $x + y \doteq x_1 + y_1$, $x - y \doteq x_1 - y_1$, $x.y \doteq x_1.y_1$. Moreover, if $\neg(y \doteq 0)$ then $\neg(y_1 \doteq 0)$ and $\frac{x}{y} \doteq \frac{x_1}{y_1}$.

The reader should formulate analogous theorems for the case that x or y is not bounded.

A rational number x is called infinitely small iff $x \doteq 0$. Evidently each infinitely small rational number is bounded.

Rational numbers x,y are said to determine the same real number iff $x \doteq y$. If we want to represent real numbers by some objects we may identify real numbers with the elements of the class BRN/$\doteq$, or use an extensional coding of BRN/$\doteq$, endowing it with appropriate algebraic operations and an ordering. We shall see later that there are some particular very useful codings of real numbers.

The results of this section show that our definition of real numbers is sound. In a limit universe, our real numbers correspond to classical real numbers since they form a linearly ordered field which contains FRN as a dense subfield and such that each class of reals having an upper bound has a least upper bound.

In a witnessed universe, real numbers are viewed as appropriate approximations of rationals, useful for computations. For example, there is no rational x such that $x^2 = 2$, but there is a rational x such that $x^2 \doteq 2$ and consequently a real x such that $x^2 = 2$. Computations with real numbers are specific computations with rational numbers characterized by the fact that infinitely small numbers are disregarded. This is how real numbers are used in all applications.

Section 3

Ordinal numbers

In Chapter I, we introduced orderings of type Ω. The axiom of two infinite cardinalities implies that the universal class V has an ordering of type Ω. Thus in our theory it suffices to work with ordinal numbers "less than Ω", i.e. ordinal numbers corresponding to countable ordinals in Cantor's set theory.

We have the same reason for introducing ordinals as in Cantor's set theory: our aim is to construct an appropriate canonical representation of orderings of type Ω. We shall define the class of ordinal numbers as a suitably chosen subclass of the class N of natural numbers.

<u>Theorem</u>. There is a $Z \subseteq N$ such that the canonical ordering $\leq$ of natural numbers restricted to Z is an ordering of type Ω and such that $\alpha < \beta$, and $\alpha, \beta \in Z$ implies $\alpha^\alpha < \beta$ for each α, β.

<u>Proof</u>. Let $\leq^*$ be an ordering of type Ω on N. Put $Z = \{\beta; (\forall \alpha <^* \beta)(\alpha^\alpha \in \beta)\}$. We prove that on Z the ordering $\leq$ and $\leq^*$ coincide. Assume $\alpha, \beta \in Z$ and $\alpha <^* \beta$. Then $\alpha^\alpha < \beta$, thus $\alpha < \beta$. Conversely, assume $\alpha < \beta$. If not $\alpha <^* \beta$ then $\beta <^* \alpha$, which implies $\beta < \alpha$, which is impossible. This proves that $\leq$ and $\leq^*$ coincide on Z. Moreover, $\alpha < \beta$ and $\alpha, \beta \in Z$ implies $\alpha^\alpha < \beta$. It suffices to show that the class Z is uncountable. Assume the contrary. Define classes Z_n by induction over FN, putting $Z_0 = Z$, $Z_{n+1} = \{\alpha^\alpha; \alpha \in Z_n\}$. For each n, Z_n is at most countable and consequently the class $\bar{Z} = \bigcup\{Z_n; n \in FN\}$ is countable. Thus $\bar{Z}$ is a semiset and $\bigcup \bar{Z}$ is a proper part of N. Let γ be the $\leq^*$-first element of $N - \bigcup \bar{Z}$. Obviously, $\gamma \notin Z$. Let $\alpha <^* \gamma$; then $\alpha \in \bigcup \bar{Z}$. Hence $\alpha \in \bigcup Z_n$ for some n; let $\beta \in Z_n$ be such that $\alpha \in \beta$. Then $\alpha^\alpha < \beta^\beta$ and $\beta^\beta \in Z_{n+1}$. This implies $\alpha^\alpha \in \bigcup \bar{Z}$ and hence $\alpha^\alpha \in \gamma$. We have proved that $\alpha <^* \gamma$ implies $\alpha^\alpha \in \gamma$, thus $\gamma \in Z$; a contradiction.

From now on, $\bar{\Omega}$ denotes a fixed class satisfying the following:

1) $FN \subseteq \bar{\Omega} \subseteq N$ and each $\alpha \in \bar{\Omega} - FN$ is even.
2) If $\alpha, \beta \in \bar{\Omega}$, $\alpha \in \beta$ and $\beta \notin FN$ then $\alpha^{\alpha} \in \beta$.
3) If $\leq$ is the natural ordering of natural numbers (i.e. $\alpha \leq \beta$ iff $\alpha \in \beta$ or $\alpha = \beta$) then $\langle \bar{\Omega}, \leq \rangle$ is an ordering of type Ω.

Starting from $\bar{\Omega}$, we shall construct another class satisfying 3) and $FN \subseteq \Omega \subseteq N$ and having some good additional properties. For each $\gamma \in \bar{\Omega}$ we define classes $p_n(\gamma)$ by induction over FN. We put $p_0(\gamma) = \{\beta;\ \beta \leq \gamma \text{ and } \beta \in \bar{\Omega}\}$, $p_{n+1}(\gamma) = \{\alpha + \beta;\ \alpha, \beta \in p_n(\gamma)\} \cup \cup \{\alpha \cdot \beta;\ \alpha, \beta \in p_n(\gamma)\}$ finally we put $p(\gamma) = \bigcup\{p_n(\gamma);\ n \in FN\}$. Evidently, for each $\alpha, \beta \in \bar{\Omega}$ such that $\alpha \in \beta$ and $\beta \notin FN$ we have $p(\alpha) \subseteq \beta$. Observe that for each $\gamma \in \bar{\Omega} - FN$ we have $p(\gamma) = \{\alpha_m \gamma^m + \ldots + \alpha_1 \gamma + \alpha_0;\ m \in FN \ \& \ \alpha_0, \ldots \alpha_m \in \bigcup\{p(\beta);\ \beta \in \gamma \text{ and } \beta \in \bar{\Omega}\}\}$.

We put $\Omega = \bigcup\{p(\gamma);\ \gamma \in \bar{\Omega}\}$ and call Ω <u>the class of all ordinal numbers</u>.

<u>Theorem</u>. $\langle \Omega, \leq \rangle$ is an ordering of type Ω. ($\leq$ is the natural ordering of natural numbers.)

<u>Proof</u>. Since $\bar{\Omega} \subseteq \Omega$, the class Ω is uncountable. Obviously, for each $\alpha \in \Omega$ the segment $\{\beta \in \Omega;\ \beta < \alpha\}$ is at most countable. Thus it suffices to show that $\leq$ is a well-ordering of Ω. For this purpose it suffices to show that the class $p(\gamma)$ is well-ordered by $\leq$ for each $\gamma \in \bar{\Omega}$. Assume the contrary and let γ be the first element of $\bar{\Omega}$ such that $p(\gamma)$ is not well-ordered. Evidently, γ is infinite. Let $\{\beta_n;\ n \in FN\}$ be a descending sequence of elements of $p(\gamma)$. Each β_n can be written in the form of a polynomial $\alpha_m \gamma^m + \ldots + \alpha_0$ where $\alpha_0, \ldots \alpha_m \in \bigcup\{p(\beta);\ \beta \in \gamma \cap \bar{\Omega}\}$; fix for each β_n one such expression. It is easy to show that one can find an descending sequence consisting of some coefficients of our countably many polynomials; but this is a contradiction.

We have $FN \subseteq \Omega \subseteq N$ and the class Ω is closed under addition and multiplication of natural numbers. One can easily check that these operations coincide for ordinal numbers with ordinal addition and multiplication defined as usual. Each limit ordinal number is even. The first element of $\Omega - FN$ (the first infinite ordinal number) is denoted by ω.

There are various possible formulations of the theorem on definition by transfinite recursion; we shall present two of them.

They will be frequently used. Since their proofs are analogous to proofs of corresponding theorems on recursion over FN, we shall only outline them.

Theorem. Let $\varphi(X,Y)$ be a formula and assume $(\forall Y)(\exists! X)\varphi(X,Y)$. Then there is exactly one relation R such that $dom(R) \subseteq \Omega$ and such that for each $\alpha \in \Omega$ we have $\varphi(R''\{\alpha\}, R \upharpoonright (\alpha \cap \Omega))$.

Proof. Let $\psi(X,\alpha)$ be the formula
$dom(X) \subseteq \alpha \cap \Omega \,\&\, Rel(X) \,\&\, \alpha \in \Omega \,\&\, (\forall\beta)(\beta \in \alpha \cap \Omega \Rightarrow \varphi(X''\{\beta\}; X\upharpoonright(\beta \cap \Omega)))$.
Put $R = \bigcup \{X; (\exists \alpha \in \Omega)\psi(X,\alpha)\}$. R has all the desired properties. The uniqueness is evident.

Theorem. Let $\varphi(x,Y)$ be a formula and assume that for each at most countable Y there is an x such that $\varphi(x,Y)$. Then there is a function F such that $dom(F) = \Omega$ and $\varphi(F(\alpha), F\upharpoonright(\alpha \cap \Omega))$ for each $\alpha \in \Omega$.

Proof. Let R be a well-ordering of V. Let $\varphi_0(x,Y)$ hold if either Y is uncountable and $x = 0$ or Y is at most countable and x is the first element such that $\varphi(x,Y)$ (w.r.t. the well-ordering R). Then $(\forall Y)(\exists! x)\ \varphi_0(x,Y)$ and for all at most countable Y, $\varphi_0(x,Y)$ implies $\varphi(x,Y)$. Let $\psi(G,\alpha)$ be the formula
$\alpha \in \Omega \,\&\, dom(G) = \alpha \cap \Omega \,\&\, (\forall \beta \in \alpha \cap \Omega)\varphi_0(G(\beta); G\upharpoonright(\beta \cap \Omega))$.
Put $F = \bigcup\{G; (\exists \alpha \in \Omega)\psi(G,\alpha)\}$; F has the desired properties.

We shall now present two examples of definition by transfinite recursion.

An operation σ associating with each countable sequence $\{X_n; n \in FN\}$ of classes from the extended universe a class $\sigma(\{X_n; n \in FN\})$ from the extended universe will be called an ω-<u>operation.</u> Clearly, a sequence $\{X_n; n \in FN\}$ is coded as the relation R such that $dom(R) \subseteq FN$ and $R''\{n\} = X_n$ for each $n \in FN$; thus an ω-operation can be understood as an operation associating with each relation R from the extended universe (satisfying $dom(R) \subseteq FN$) a class from the extended universe.

Let $\mathcal{M}$ be a codable class and let σ be an ω-operation. $\mathcal{M}$ is said to be closed under σ if for each sequence $\{X_n; n \in FN\}$ of elements of $\mathcal{M}$ the class $\sigma(\{X_n; n \in FN\})$ belongs to $\mathcal{M}$.

Theorem. Let $\mathcal{N}$ be a codable class and let σ be an ω-operation. Then there is a codable class $\mathcal{M}$ such that

1) $\mathcal{N} \subseteq \mathcal{M}$;

2) $\mathcal{M}$ is closed under σ;

3) if $\mathcal{M}_0$ is a codable class satisfying 1) and 2) then $\mathcal{M} \subseteq \mathcal{M}_0$.

Proof. Let $\langle \bar{K},\bar{S} \rangle$ code the class V^{FN} of all countable sequences of sets. We construct sequences $\{K_\alpha ; \alpha \in \Omega\}$ and $\{S_\alpha ; \alpha \in \Omega\}$ by recursion over Ω. Let $\langle K_0,S_0 \rangle$ be a coding of $\mathcal{N}$. Let now $\gamma \neq 0$, $\gamma \in \Omega$. Put $\bar{\bar{K}} = \{\langle x, \alpha \rangle ; \alpha \in \gamma \cap \Omega \ \& \ x \in K_\alpha\}$ and $\bar{\bar{S}} = \{\langle y, \langle x, \alpha \rangle\rangle ; \langle x, \alpha \rangle \in \bar{\bar{K}} \ \& \ y \in S_\alpha " \{x\}\}$. Furthermore, put $K_\gamma = \{x ; x \in \bar{K} \ \& \ \bar{S}"\{x\} \in \bar{\bar{K}}^{FN}\}$. Thus $\langle K_\gamma , \bar{S} \rangle$ codes $\bar{\bar{K}}^{FN}$. Put $\bar{\bar{\bar{S}}} = \{\langle w,x \rangle ; x \in K_\gamma \ \& \ (\exists z,y,n)(\langle z,n \rangle \in \bar{S}"\{x\} \ \& \ y \in \bar{\bar{S}}"\{z\} \ \& \ w = \langle y,n \rangle)\}$. Thus $\langle K_\gamma , \bar{\bar{\bar{S}}} \rangle$ codes the class of all relations R such that $\mathrm{dom}(R) \subseteq FN$ and $(\forall n)(\exists z)(R"\{n\} = \bar{\bar{S}}"\{z\})$. Finally we put $S_\gamma = \{\langle y,x \rangle ; x \in K_\gamma \ \& \ y \in \sigma(\bar{\bar{\bar{S}}}"\{x\})\}$. Thus $\langle K_\gamma , S_\gamma \rangle$ codes the class of all classes $\sigma(R)$ where R is as above. Having constructed $\langle K_\alpha , S_\alpha \rangle$ for each $\alpha \in \Omega$ we put

$K = \{\langle x, \alpha \rangle ; \alpha \in \Omega \ \& \ x \in K_\alpha\}$,

$S = \{\langle y, \langle x, \alpha \rangle\rangle ; \langle x, \alpha \rangle \in K \ \& \ y \in S_\alpha " \{x\}\}$. The class $\mathcal{M}$ coded by $\langle K,S \rangle$ has the desired properties.

The class $\mathcal{M}$ is determined uniquely by $\mathcal{N}$; we shall call $\mathcal{M}$ the closure of $\mathcal{N}$ w.r.t. σ. The reader can easily generalize the above construction for the case of several ω-operations.

We shall now prove some theorems enabling us to transfer easily some results of Cantor's set theory into our theory.

Theorem. There is a relation S such that $\langle \Omega ,S \rangle$ codes extensionally $P_\omega(\Omega)$ and $S"\{\alpha\} \subseteq \alpha$ for each $\alpha \in \Omega$.

Proof. Let F be a one-one mapping of Ω onto V. By the axiom of prolongation, for each at most countable class $X \subseteq \Omega$ the class $\{y ; \Omega \cap y = X\}$ is uncountable. Thus $\{\alpha ; \Omega \cap F(\alpha) = X\}$ is uncountable. We define a class $Z \subseteq \Omega$ by induction over Ω as follows: for $\gamma \in \Omega$, $\gamma \in Z$ iff $F(\gamma) \cap \Omega \subseteq \gamma \cap Z$ and $(\forall \alpha \in \gamma \cap \Omega)(F(\alpha) \cap \Omega \neq F(\gamma) \cap \Omega)$. Put $S_1 = \{\langle \beta , \alpha \rangle ; \alpha \in Z \ \& \ \beta \in F(\alpha) \cap \Omega\}$. Obviously, $S_1"\{\alpha\} = F(\alpha) \cap \Omega = F(\alpha) \cap Z$ for each $\alpha \in Z$. If α , β are distinct elements of Z then $S_1"\{\alpha\} \neq S_1"\{\beta\}$. If $X \subseteq Z$ is an at most countable class then let γ be the first ordinal number such that $X \subseteq \gamma$ and $F(\gamma) \cap \Omega = X$. Then $\gamma \in Z$. Consequently $\langle Z, S_1 \rangle$ codes extensionally $P_\omega(Z)$ and Z is uncountable. Let G be a one-one mapping of Ω onto Z preserving the natural ordering $\leq$. Put $S = \{\langle \alpha , \beta \rangle ; \langle G(\alpha), G(\beta) \rangle \in S_1\}$.

Then $\langle \Omega, S \rangle$ has all the desired properties.

Theorem. There is a class A satisfying the following:

1) $(\forall x,y \in A)(x = y \text{ iff } x \cap A = y \cap A)$;
2) $(\forall x \in A)(x \cap A$ is at most countable$)$;
3) $(\forall X \subseteq A)(X$ at most countable $\Rightarrow (\exists x \in A)(X = x \cap A))$;
4) $(\forall x \subseteq A)(x \in A)$.

Proof. Let $\langle \Omega, S \rangle$ be as in the preceding theorem. Obviously, $S''\{0\} = \emptyset$. Let G be a one-one mapping of Ω onto V. We construct a function F by induction. Let $\alpha \in \Omega$. If $S''\{\alpha\}$ is a finite set then put $F(\alpha) = F''S''\{\alpha\}$. If $S''\{\alpha\}$ is countable then there is a $\gamma \in \Omega$ such that $F''S''\{\alpha\} \subseteq G(\gamma)$, $G(\gamma) \cap F''(\alpha - S''\{\alpha\}) = \emptyset$, $\tau(G(\gamma)) > \tau(x)$ for each $x \in F''\alpha$, $G(\gamma) \cap (P_n(F''\alpha) - F''S''\{\alpha\}) = \emptyset$ for each n, where $P_n(X) = P(P(\ldots(P(X))\ldots))$ (n time). In this case we put $F(\alpha) = G(\gamma)$ where γ is the first ordinal number such that $G(\gamma)$ is as above. It suffices to put $A = F''\Omega$; A has all desired properties.

The class A guaranteed by the preceding theorem can be viewed as a model of the Cantorian universe of hereditarily countable sets. This enables us to translate all results concerning the Cantorian universe of hereditarily countable sets into results concerning the class A. Of particular interest are results concerning subclasses of FN and not mentioning A. It is easily seen that A is not determined uniquely.

Section 4

Ultrafilters

The notion of an ultrafilter is a typical example of a notion whose importance is not confined to Cantor's set theory. An ultrafilter can be understood as splitting a family of properties into two mutually dual parts.

We shall deal with codable classes in order to achieve the greatest generality; this is necessary because of the general nature of the notions we are going to study. On the other hand, we shall study ultrafilters only on rings of classes, which simplifies some formulations.

German letters $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{R}$, $\mathfrak{J}$, possibly indexed, will be used as variables for codable classes. The meaning of expressions $\mathfrak{M} \subseteq \mathfrak{N}$, $\mathfrak{M} = \mathfrak{M}_1 \cap \mathfrak{M}_2$, etc. is clear.

$\mathfrak{R}$ is called a <u>ring of classes</u> iff

(1) $\bigcup\mathfrak{R}$ is an infinite class and $\bigcup\mathfrak{R} \in \mathfrak{R}$,

(2) $(\forall X)(\forall Y)(X,Y \in \mathfrak{R} \Rightarrow X \cap Y \in \mathfrak{R})$,

(3) $(\forall X)(X \in \mathfrak{R} \Rightarrow (\bigcup\mathfrak{R} - X) \in \mathfrak{R})$, and

(4) if $X \in \mathfrak{R}$ and X has at least two elements then there is a proper non-empty subclass Y of X such that $Y \in \mathfrak{R}$.

Let us mention at least the following examples: $P(x)$ for each infinite set x, $V \cup \{V - x;\ x \in V\}$, $P_\omega(X)$ for each countable class, etc.

In the whole section, $\mathfrak{R}$ denotes a ring of classes.

<u>Theorem</u>. If $X,Y \in \mathfrak{R}$ then $X \cup Y \in \mathfrak{R}$ and $X - Y \in \mathfrak{R}$.

<u>Proof</u>. $X - Y = X \cap (\bigcup\mathfrak{R} - Y)$; $X \cup Y = \bigcup\mathfrak{R} - ((\bigcup\mathfrak{R} - X) \cap (\bigcup\mathfrak{R} - Y))$.

Note that we obtain $\emptyset \in \mathfrak{R}$ as an immediate consequence.

A class $\mathfrak{M}$ is a <u>filter</u> on $\mathfrak{R}$ iff

(1) $\mathcal{M}$ is a non-empty proper subclass of $\mathcal{R}$;

(2) $(\forall X,Y \in \mathcal{M})(X \cap Y \in \mathcal{M})$, i.e. $\mathcal{M}$ is closed under intersection;

(3) $(\forall X \in \mathcal{M})(\forall Y \in \mathcal{R})(X \subseteq Y \Rightarrow Y \in \mathcal{M}$, i.e. $\mathcal{M}$ is closed under taking superclasses.

$\mathcal{M}$ is an <u>ultrafilter</u> iff, in addition,

(4) $(\forall X \in \mathcal{R})(X \in \mathcal{M} \vee (\bigcup\mathcal{R} - X) \in \mathcal{M})$, i.e. either X or the complement of X belongs to $\mathcal{M}$, for each $X \in \mathcal{R}$.

Note that if $\mathcal{M}$ is a filter on $\mathcal{R}$ then $\emptyset \notin \mathcal{M}$.

<u>Theorem</u>. Let $\mathcal{M} \subseteq \mathcal{R}$, $\mathcal{M} \neq \emptyset$, $\emptyset \notin \mathcal{M}$, and let $\mathcal{M}$ be dually directed by inclusion. Then $\{X \in \mathcal{R};\ (\exists Y \in \mathcal{M})(Y \subseteq X)\}$ is a filter on $\mathcal{R}$.

The following theorem shows that an ultrafilter can be viewed as a partition of $\mathcal{R}$ into two dual parts.

<u>Theorem</u>. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two disjoint classes whose union is $\mathcal{R}$. Assume that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are closed under intersection and let, for each $X \in \mathcal{R}$, $X \in \mathcal{M}_1$ iff $(\bigcup\mathcal{R} - X) \in \mathcal{M}_2$. Then either $\mathcal{M}_1$ or $\mathcal{M}_2$ is an ultrafilter on $\mathcal{R}$.

<u>Proof</u>. Assume $\bigcup\mathcal{R} \in \mathcal{M}_1$; then $\emptyset \in \mathcal{M}_2$. We prove that $\mathcal{M}_1$ is an ultrafilter on $\mathcal{R}$. The conditions (1), (2), (4) hold evidently for $\mathcal{M}_1$. We prove (3). Assume $X \in \mathcal{M}_1$, $Y \in \mathcal{R}$, and $X \subseteq Y$. If $Y \notin \mathcal{M}_1$ then $(\bigcup\mathcal{R} - Y) \in \mathcal{M}_1$ and hence $\emptyset = X \cap (\bigcup\mathcal{R} - Y) \in \mathcal{M}_1$, a contradiction. Thus $Y \in \mathcal{M}_1$, which concludes the proof.

<u>Theorem</u>. For each filter $\mathcal{N}$ on $\mathcal{R}$, there is an ultrafilter $\mathcal{M}$ on $\mathcal{R}$ such that $\mathcal{N} \subseteq \mathcal{M}$.

<u>Proof</u>. If $\mathcal{R}$ is uncountable then it has an enumeration $\{X_\alpha;\ \alpha \in \Omega\}$; if it is countable then it has an enumeration $\{X_\alpha;\ \alpha \in FN\}$. Thus let $\{X_\alpha;\ \alpha \in A\}$ be such an enumeration where A is either Ω or FN. We construct a class $A_0 \subseteq A$ by recursion. For each $\gamma \in A$, we put $\gamma \in A_0$ iff for each $X \in \mathcal{N}$ and each set $v \subseteq \gamma \cap A_0$ we have $X_\gamma \cap X \cap \bigcap \{X_\alpha,\ \alpha \in v\} \neq \emptyset$. We now define $\mathcal{M} = \{X;\ (\exists \alpha \in A_0)(X = X_\alpha)\}$. Evidently $\mathcal{N} \subseteq \mathcal{M}$; the reader can easily verify that $\mathcal{M}$ is an ultrafilter on $\mathcal{R}$.

An ultrafilter $\mathcal{M}$ on $\mathcal{R}$ is <u>trivial</u> iff there is a y such that $\{y\} \in \mathcal{M}$. Evidently $\mathcal{M}$ is trivial iff there is a finite set u such that $u \in \mathcal{M}$.

Theorem. Let $\mathcal{R}$ be a filter on $\mathcal{R}$ not containing any finite set. Then there is a non-trivial ultrafilter $\mathcal{M}$ on $\mathcal{R}$ such that $\mathcal{R} \subseteq \mathcal{M}$.

Proof. Let $\mathcal{R}_0$ consist of all classes of the form $(\bigcup \mathcal{R} - u)$, where u is a finite set and $u \in \mathcal{R}$. Put $\mathcal{R}_1 = \{Z \in \mathcal{R};\ (\exists X \in \mathcal{R}_0)(\exists Y \in \mathcal{R})(X \cap Y \subseteq Z)\}$. Then $\mathcal{R} \subseteq \mathcal{R}_1$ and $\mathcal{R}_1$ is a filter on $\mathcal{R}$. If $\mathcal{M}$ is an ultrafilter on $\mathcal{R}$ extending $\mathcal{R}_1$ then $\mathcal{M}$ cannot contain any finite set.

A class $\mathcal{J}$ is an ideal on $\mathcal{R}$ if

(1′) $\mathcal{J}$ is a non-empty proper subclass of $\mathcal{R}$;

(2′) $(\forall X,Y \in \mathcal{J})(X \cup Y \in \mathcal{J})$, i.e. $\mathcal{J}$ is closed under union;

(3′) $(\forall X \in \mathcal{J})(\forall Y \in \mathcal{R})(Y \subseteq X \Rightarrow Y \in \mathcal{J})$, i.e. $\mathcal{J}$ is closed under taking subclasses.

$\mathcal{J}$ is a prime ideal if, in addition,

(4′) $(\forall X \in \mathcal{R})(X \in \mathcal{J} \vee (\bigcup \mathcal{R} - X) \in \mathcal{J})$.

The duality between filters and ideals is so obvious that results on filters can be immediately translated into results on ideals and vice versa. Therefore we shall not consider ideals in the present section.

A countable sequence $\{X_n;\ n \in FN\}$ is descending iff $X_{n+1} \subseteq X_n$ for each n. A class $\mathcal{M}$ is ω-complete (from above) iff $\mathcal{M}$ is non-empty, no element of $\mathcal{M}$ is a finite set, and for each descending sequence $\{X_n;\ n \in FN\}$ of elements of $\mathcal{M}$ there is an $X \in \mathcal{M}$ such that $X \subseteq \bigcap\{X_n;\ n \in FN\}$.

Observe that each ω-complete ultrafilter $\mathcal{M}$ on $\mathcal{R}$ is non-trivial.

Theorem. Let $\mathcal{R}$ be an ω-complete subclass of $\mathcal{R}$ such that for each infinite $X \in \mathcal{R}$ there is a $Y \in \mathcal{R}$ such that $Y \subseteq X$. Then there is an ω-complete ultrafilter $\mathcal{M}$ on $\mathcal{R}$.

Proof. Obviously, $\mathcal{R}$ is uncountable. Let $\{X_\alpha;\ \alpha \in \Omega\}$ be an enumeration of elements of $\mathcal{R}$. We construct a sequence $\{Y_\alpha;\ \alpha \in \Omega\}$ by recursion as follows: Y_α is the first element of $\mathcal{R}$ (w.r.t. to the given enumeration) such that $Y_\alpha \subseteq \bigcap\{Y_\beta;\ \beta \in \alpha \cap \Omega\}$ and either $Y_\alpha \subseteq X_\alpha$ or $Y_\alpha \cap X_\alpha = \emptyset$. Obviously, Y_α exists for each $\alpha \in \Omega$. Put $\mathcal{M} = \{X \in \mathcal{R};\ (\exists \alpha \in \Omega)(Y_\alpha \subseteq X)\}$. It is easy to verify that $\mathcal{M}$ is an ω-complete ultrafilter on $\mathcal{R}$.

The axiom of prolongation implies that the class of all infinite sets is ω-complete. Thus we have the following.

Corollary. Let x be an infinite set. Then there is an ω-complete ultrafilter Z on the ring P(x) of all subsets of x.

Section 5

Basic languages

The language of mathematics is one of the traditional mathematical structures and has been frequently subjected to mathematical investigation. In the present section we are going to construct basic languages for a description of the extended universe. The construction proceeds inside our theory.

We use formulas to express some properties of sets and classes. Formulas of a certain language are particular sequences of symbols of an alphabet, formed according to some exact syntactic rules. In each particular case we could deal with a finite alphabet; nevertheless, it is convenient to use a potentially infinite alphabet.

Our alphabet consists of the following systems of signs:

(1) Specific symbols $\in$, $=$, $\&$, $\vee$, $\Rightarrow$, $\equiv$, $\neg$, $\exists$, $\forall$,), (.

(2) Variables for sets from the universe of sets: x_0, x_1, ...

(3) Variables for classes from the extended universe: X_0, X_1, ...

(4) Constants for sets from the universe of sets.

Since our aim is to construct the language inside the universe of sets, we shall proceed analogously to the construction of natural numbers. Some sets will be identified with signs. This can be achieved in various ways; we choose the following:

(1) The sets $\langle 0,0 \rangle$, $\langle 1,0 \rangle$, ..., $\langle 10,0 \rangle$ are (code) the signs $\in$, $=$, $\&$, $\vee$, $\Rightarrow$, $\equiv$, $\neg$, $\exists$, $\forall$,), (, respectively.

(2) $\langle \alpha,1 \rangle$ is (codes) the variable x_α, for each $\alpha \in N$.

(3) $\langle \alpha,2 \rangle$ is (codes) the variable X_α, for each $\alpha \in N$.

(4) $\langle x,3 \rangle$ is (codes) the constant denoting x, for each $x \in V$.
We put Const $= \{\langle x,3 \rangle ; x \in V\}$.

The alphabet Alph is the class consisting of all sets (1) - (4). A <u>word</u> is a function f such that $\mathrm{dom}(f) \in N$ and $\mathrm{rng}(f) \subseteq$ Alph. The class of all words is denoted by Word.

From now on, we shall freely use the original signs and sequences of signs since the coding by sets will always be immediate.

Let C be a class.

Formulas of the language L_C are words obtainable by the following rules:

(1) Atomic formulas of L_C are words $\Gamma \in \Delta$, $\Gamma = \Delta$, $\Gamma \in X_i$ where Γ, Δ are variables or constants for elements of C.

(2) If φ and ψ are formulas of L_C and Γ is a variable then the following words are formulas of L_C:
$(\varphi \& \psi)$, $(\varphi \vee \psi)$, $(\varphi \Rightarrow \psi)$, $(\varphi \equiv \psi)$, $\neg(\varphi)$, $(\exists \Gamma)\varphi$, $(\forall \Gamma)\varphi$.

The class of all formulas of L_C will be denoted by L_C. It can be defined using recursion over N; thus L_C is set-theoretically definable.

A formula φ of L_C is normal if no class variable is quantified in φ; φ is a set-formula if no class variable occurs in φ.

L is the class of formulas containing no constants. Obviously, L is set-theoretically definable, $L = L_{\emptyset}$.

We shall freely use common symbolism. For example denoting a formula of L_C by $\varphi(x_0,x_1)$ we mean implicitly that all variables distinct from x_0, x_1 and occuring in φ are quantified in φ (i.e. φ has no free variables distinct from x_0, x_1). If $\varphi(x_0,x_1)$ is a formula and if q is a constant then $\varphi(x_0,q)$ denotes the formula resulting from φ by replacing all free occurences of x_1 by q, etc.

Like natural numbers, formulas have been constructed inside the universe of sets. From now on, we shall investigate the language L_C from the point of view of the extended universe.

FL_C denotes the class (language) of all formulas of L_C such that

(1) φ is a finite set (a finite sequence);

(2) If the variable x_α or X_α occurs in φ then $\alpha \in FN$.

We further put $FL = FL_{\emptyset}$. Evidently, FL is a countable class.

We already know that some properties of sets (from the universe of sets) can be expressed by set-formulas of L_C. Let us now ask the converse question, namely, which formulas of L_C can be recognized as expressing some properties of sets. The question is, which formulas can be read, i.e. understood. We can read atomic formulas; if we can read φ and ψ then we can read each formula formed from them using rule (2) of the definition of formulas. Thus each formula of FL_C can be understood. It is possible that also some other formulas can be understood; but we cannot be sure at present. Later we shall see that the axiom of prolongation makes such a hypothesis possible.

We shall now be more explicit on the notion of set-theoretically definable classes. From now on, a class X is called set-theoretically definable (notation: Sd(X)) iff there is a set-formula $\varphi(x_0) \in FL_V$ such that $X = \{x_0; \varphi(x_0)\}$. Evidently, each set is a set-theoretically definable class. The following theorem shows our definition of set-theoretically definable classes is sound:

Theorem. The class $\{X; Sd(X)\}$ is codable.

Proof. Let K be the class of all formulas $\varphi(x_0) \in FL_V$ with one free variable. We define a relation S with $dom(S) \subseteq K$ by induction over FN as follows: For atomic φ, we put $S''\{\varphi(x_0)\} = \{x_0; \varphi(x_0)\}$. If φ is $\varphi_1(x_0)$ & $\varphi_2(x_0)$ then we put $S''\{\varphi(x_0)\} = S''\{\varphi_1(x_0)\} \cap S''\{\varphi_2(x_0)\}$; if φ is $\neg\varphi_1(x_0)$ then we put $S''\{\varphi(x_0)\} = V - S''\{\varphi_1(x_0)\}$. Similarly for other connectives. If $\varphi(x_0)$ is $(\exists x_1)\ \varphi_1(x_0, x_1)$ then we put $S''\{\varphi(x_0)\} = \bigcup\{S''\{\varphi_1(x_0, q)\}; q \in Const\}$. Similarly for $\forall$. A pedantical definition proceeds by recursion on the number of quantifiers in φ. The coding pair $\langle K,S\rangle$ codes $\{X; Sd(X)\}$.

Now we have to reformulate the axiom of induction in such a way that it admits exactly set-theoretically definable classes. We obtain the following formulation:

(Axiom of induction.)

$$(\forall X)(Sd(X) \ \&\ \emptyset \in X \ \&\ (\forall x \in X)(\forall y)(x \cup \{y\} \in X) \Rightarrow X = V).$$

It can be seen that this reformulation does not affect any particular results obtained till now. The reformulation would have to be taken into account if the theory were formalized.

- x - X - x -

A class X is called revealed if for each countable $Y \subseteq X$ there is a set u such that $Y \subseteq u \subseteq X$.

Theorem. Let $X \cap u$ be a set for each set u. Then X is revealed.

Proof. Let $Y \subseteq X$, Y countable. There is a set v such that $Y \subseteq v$. Put $u = X \cap v$. Then $Y \subseteq u \subseteq X$.

Corollary. Each set-theoretically definable class is revealed.

Theorem. (1) If X, Y are revealed then $X \cup Y$ is revealed.
(2) If $\{X_n;\ n \in FN\}$ is a sequence of revealed classes then $\bigcap \{X_n;\ n \in FN\}$ is a revealed class.

Proof of (2). Let Y be a countable subclass of $\bigcap\{X_n;\ n \in FN\}$. There is a sequence $\{u_n;\ n \in FN\}$ of sets such that $Y \subseteq u_n \subseteq X_n$ for each n. Thus $Y \subseteq \bigcap\{u_n;\ n \in FN\}$. By a consequence of the axiom of prolongation, there is a set u such that $Y \subseteq u \subseteq \bigcap\{u_n;\ n \in FN\}$.

Theorem. Let $\{X_n;\ n \in FN\}$ be a sequence of revealed classes such that for each $m \in FN$, $\bigcap\{X_n;\ n \leq m\}$ is non-empty. Then $\bigcap\{X_n;\ n \in FN\} \neq \emptyset$.

Proof. For each n, choose $x_n \in X_0 \cap \ldots \cap X_n$. Put $Y_n = \{x_m;\ n \leq$ Each Y_n is at most countable and we have $Y_n \subseteq X_n$ for each n. Thus there is a sequence $\{u_n;\ n \in FN\}$ of sets such that $Y_n \subseteq u_n \subseteq X_n$ for each n. Since $x_n \in u_0 \cap \ldots \cap u_n$ each finite subcollection of $\{u_n;\ n \in FN\}$ has non-empty intersection. The axiom of prolongation implies $\bigcap\{u_n;\ n \in FN\} \neq \emptyset$. But $\bigcap\{u_n;\ n \in FN\} \subseteq \bigcap\{X_n;\ n \in FN\}$.

Theorem. Let $\{X_n;\ n \in FN\}$ be a descending sequence of revealed classes. Put $X = \bigcap\{X_n;\ n \in FN\}$. Then $dom(X) = \bigcap\{dom(X_n);\ n \in FN\}$.

Proof. Obviously, $dom(X) \subseteq \bigcap\{dom(X_n);\ n \in FN\}$. Let $x \in \bigcap\{dom(X_n);\ n \in FN\}$. Then $\{X_n \cap (V \times \{x\});\ n \in FN\}$ is a descendi sequence of revealed non-empty classes and hence $\emptyset \neq \bigcap\{X_n \cap (V \times \{x\});\ n \in FN\} = X''\{x\}$.

A class X is a σ-class (a π-class) if X is the union (the intersection) of a countable sequence of set-theoretically definable classes. We can analogously define $\sigma\pi$-classes, $\pi\sigma$-classes etc.

Theorem.
(1) The class of all σ-classes and the class of all π-classes ar codable.
(2) X is a σ-class iff X is the union of a countable ascending sequence of set-theoretically definable classes. X is a π-cla iff X is the intersection of a countable descending sequence o set-theoretically definable classes.

(3) The union of two π-classes is a π-class. The intersection of two σ-classes is a σ-class.

(4) The union of a countable sequence of σ-classes is a σ-class. The intersection of a countable sequence of π-classes is a π-class.

(5) If X, Y are σ-classes (π-classes) then $X \times Y$ is a σ-class (a π-class).

(6) Each π-class is revealed.

(7) If X is a σ-class (a π-class) then $\mathrm{dom}(X)$ is a σ-class (a π-class).

Proof. (1) - (5) are obvious. (6) Each set-theoretically definable class is revealed and revealed classes are closed under countable intersection. (7) The assertion is obvious for σ-classes. If X is a π-class then X is an intersection of countably many revealed classes X_n; $n \in FN$; and for such classes we have proved $\mathrm{dom}(\bigcap\{X_n;\ n \in FN\}) = \bigcap\{\mathrm{dom}(X_n);\ n \in FN\}$.

Theorem. X is both a σ-class and a π-class iff X is set-theoretically definable.

Proof. The implication $\Leftarrow$ is obvious; we prove $\Rightarrow$. Let X be both a σ-class and a π-class. Then X is revealed and is the union of an ascending sequence $\{X_n;\ n \in FN\}$ of set-theoretically definable classes. We prove that $X = X_n$ for some n. Assume that X_n is a proper subclass of X_{n+1} for each n. Let $\{y_n;\ n \in FN\}$ be a sequence such that $y_n \in X_{n+1} - X_n$ for each n. Let v be a set such that $v \subseteq X$ and $y_n \in v$ for each n. Then $v \cap (V - X_n) \neq \emptyset$ for each n; but since each class $v \cap (V - X_n)$ is revealed we have $v \cap \bigcap\{V - X_n;\ n \in FN\} \neq \emptyset$, hence $v \cap (V - X) \neq \emptyset$, which contradicts $v \subseteq X$.

Chapter III

Topological shapes

One of the most important aims of mathematics is to master continuum phenomena. If we observe a set but are unable to identify (distinguish) its individual elements, because they lie beyond the horizon of our observational capability, we encounter a continuum phenomenon. Thus, for example, when we observe a heap of sand from a sufficient distance it appers to be continuous.

When treating continuum phenomena mathematically we shall accept the hypothesis that all continuum phenomena are produced by such observation of large but remote sets or classes. This hypothesis is in accordance with the aims of set theory.

We conclude that continuum phenomena are due to the indiscernibility of particular elements of the observed class. The relation of indiscernibility is evidently reflexive and symmetric. In the present chapter we confine ourselves to transitive relations of indiscernibility, hence to equivalences. This enables us to treat continuum phenomena classically. This is achieved by decomposing a continuum phenomenon into points - specific codes of mutually indiscernible elements of the observed class.

One characteristic property of a set is its shape. The shape of a set depends indeed on the method of observation, thus on the corresponding equivalence relation of indiscernibility. For example, if we observe a book (consisting of molecules) we perceive a shape entirely different from the shape perceived by an observer the size of a molecule. Needless to say, we do not restrict ourselves to optical observation.

The present chapter is devoted to the study of properties of shapes determined only by the equivalence relation of indiscernibility, thus to topological problems. But we shall not restrict ourselves to shapes of sets and shall investigate also shapes of classes.

Section 1

Equivalences of indiscernibility

The mathematical counterpart of the notion of indiscernibility is an equivalence relation having some additional properties. Each observation produces a sequence of criteria of discernibility. Two objects are indiscernible under such an observation if all criteria fail to distinguish between them. This leads us to the requirement that an equivalence of indiscernibility should be the intersection of countably many set-theoretically definable classes.

A class $\doteq$ is a <u>π-equivalence</u> if $\doteq$ is a π-class and an equivalence relation. A sequence $\{R_n;\ n \in FN\}$ is a <u>generating sequence</u> of an equivalence $\doteq$ iff the following conditions hold:

(1) For each n, R_n is a set-theoretically definable, reflexive, and symmetric relation.

(2) For each n and each x, y, z, $\langle x,y\rangle \in R_{n+1}$ and $\langle y,z\rangle \in R_{n+1}$ implies $\langle x,z\rangle \in R_n$; $R_o = V^2$.

(3) $\doteq$ is the intersection of all the classes R_n.

If $\{R_n;\ n \in FN\}$ is a generating sequence of $\doteq$ then $\doteq$ is a π-equivalence, $R_{n+1} \subseteq R_n$ for each n, and $x \doteq y$ holds iff $\langle x,y\rangle \in R_n$ for each n.

<u>Theorem</u>. Each π-equivalence has a generating sequence.

<u>Proof</u>. Let $\doteq$ be a π-equivalence. Then $\doteq$ is the intersection of countably many set-theoretically definable classes $\{X_n;\ n \in FN\}$. We can assume without loss of generality that the last sequence is descending. Put $S_n = X_n \cap X_n^{-1}$. Then $\{S_n;\ n \in FN\}$ satisfies everything we have claimed for $\{X_n;\ n \in FN\}$ and in addition, each relation S_n is reflexive and symmetric. We claim that for each m there is an $n > m$ such that for each x, y, z, $\langle x,y\rangle \in S_n$ and $\langle y,z\rangle \in S_n$ implies $\langle x,z\rangle \in S_m$. Assume the contrary and let, for each $n > m$, x_n, y_n, z_n be such that $\langle x_n,y_n\rangle \in S_n$, $\langle y_n,z_n\rangle \in S_n$ but $\langle x_n,z_n\rangle \notin S_m$. By the axiom of prolongation, there are x, y, z such that $\langle x,z\rangle \notin S_m$

but $\langle x,y \rangle \in S_n$ and $\langle y,z \rangle \in S_n$ for each $n > m$. Thus $x \doteq y$ and $y \doteq z$, hence $x \doteq z$ and $\langle x,z \rangle \in S_m$, which is a contradiction. This proves our claim. Consequently, there is a subsequence $\{R_n;\ n \in FN\}$ of $\{S_n;\ n \in FN\}$ such that $\bigcap\{R_n;\ n \in FN\} = \bigcap\{S_n;\ n \in FN\}$ and for each n and each x, y, z, $\langle x,y \rangle \in R_{n+1}$ and $\langle y,z \rangle \in R_{n+1}$ implies $\langle x,z \rangle \in R_n$. This concludes the proof.

Let $\{R_n;\ n \in FN\}$ be a generating sequence of $\doteq$ and let u be a set. A sequence $\{r_\alpha;\ \alpha \leq \gamma\}$ is a <u>prolongation</u> of $\{R_n;\ n \in FN\}$ <u>on</u> u if the following conditions hold:

(0) $\{r_\alpha;\ \alpha \leq \gamma\}$ is a set and $\gamma \in N - FN$.

(1) For each $\alpha \leq \gamma$, r_α is a reflexive and symmetric relation on u.

(2) For each $\alpha < \gamma$ and each x, y, z, $\langle x,y \rangle \in r_{\alpha+1}$ and $\langle y,z \rangle \in r_{\alpha+1}$ implies $\langle x,z \rangle \in r_\alpha$.

(3) $r_n = R_n \cap u^2$ for each n.

(4) $r_\gamma = \{\langle x,x \rangle;\ x \in u\}$.

The following theorem is a direct consequence of the axiom of prolongation:

<u>Theorem</u>. Let $\{R_n;\ n \in FN\}$ be a generating sequence of $\doteq$. Then for each u there is a sequence $\{r_\alpha;\ \alpha \leq \gamma\}$ prolonging the former sequence on u.

We shall formulate a further condition imposed on equivalences of indiscernibility. It results from the following consideration: no infinite set lies before the horizon. Thus each infinite set of observed objects has at least one pair of mutually indiscernible elements.

An equivalence $\doteq$ is said to be <u>compact</u> if for each infinite set u there are $x,y \in u$ such that $x \neq y$ and $x \doteq y$.

Let R be a symmetric relation. A class X is an R-net iff there are no distinct elements $x,y \in X$ such that $\langle x,y \rangle \in R$. X is a <u>maximal</u> R-net <u>on</u> Z if $X \subseteq Z$ and for each $z \in Z$ there is an $x \in X$ such that $\langle x,z \rangle \in R$.

Evidently, each subclass of an R-net is an R-net.

A relation R is an <u>upper bound</u> of an equivalence $\doteq$ if R is symmetrical, set-theoretically definable and $\doteq$ is a subclass of R, i.e. $x \doteq y$ implies $\langle x,y \rangle \in R$ for each x,y.

Theorem. Let $\doteq$ be a compact equivalence and let R be its upper bound. Then there is a finite number n such that for each R-net X we have $X \precsim n$.

Proof. Assume that for each n there is an R-net u having exactly n elements. By the axiom of prolongation, there is an $\alpha \in N - FN$ and an R-net u such that $\alpha \hat{\approx} u$. Thus u is infinite and there are $x,y \in u$ such that $x \neq y$ and $x \doteq y$. This implies $\langle x,y \rangle \in R$ and hence $x = y$, which is a contradiction.

Corollary. Under the assumptions of the preceding theorem, each R-net is a set.

Theorem. Let $\doteq$ be a compact equivalence and let R be its upper bound. Then for each Z there is a set u which is a maximal R-net on Z.

Proof. Take an R-net $u \subseteq Z$ having the maximal possible number of elements.

Theorem. Let $\{R_n;\ n \in FN\}$ be a generating sequence of an equivalence $\doteq$. Let $\{u_n;\ n \in FN\}$ be a sequence of sets such that for each n u_n is a maximal R_n-net on X. If $\bigcup \{u_n;\ n \in FN\} \subseteq u$ then for each $x \in X$ there is a $y \in u$ such that $x \doteq y$.

Proof. Let $x \in X$. There is a sequence $\{y_n;\ n \in FN\}$ such that, for each n, $y_n \in u_n$ and $\langle x,y_n \rangle \in R_n$. We have $y_n \in u$ for each n; by the axiom of prolongation, there is a $y \in u$ such that $\langle x,y \rangle \in R_n$ for each n, hence $x \doteq y$.

Theorem. Let $\doteq$ be a π-equivalence. Then the following properties are equivalent:

(1) $\doteq$ is compact;

(2) for each $\gamma \in N - FN$, there is a set u such that $u \hat{\precsim} \gamma$ and for each x there is a $y \in u$ such that $x \doteq y$;

(3) each infinite set u has an infinite subset v such that for each $x,y \in v$ we have $x \doteq y$.

Proof. Let $\{R_n;\ n \in FN\}$ be a generating sequence of $\doteq$. The implication (3) => (1) is trivial. We prove (1) => (2). Let $\{u_n;\ n \in FN\}$ be a sequence of finite sets such that for each n u_n is a maximal R_n-net on V. Put $Y = \bigcup \{u_n;\ n \in FN\}$. Y is at most countable. Consequently, for each infinite γ there is a set u such that $Y \subseteq u$ and $u \hat{\precsim} \gamma$. By the preceding theorem, for each x there is a $y \in u$ such that $x \doteq y$.

(2) $\Rightarrow$ (3). Let u be an infinite set. Take a $\gamma \notin FN$ such that $\gamma^2 \hat{\lesssim} u$. Let w be such that $w \hat{\lesssim} \gamma$ and $(\forall x)(\exists y \in w)(x \doteq y)$. Let $\{r_\alpha;\ \alpha \le \tau\}$ be a prolongation of $\{R_n;\ n \in FN\}$ on $u \cup w$. For $x \in u$ let $g(x)$ be the maximal $\alpha \le \tau$ such that $w \cap \{y \in u \cup w;\ \langle x,y\rangle \in r_\alpha\} \neq \emptyset$. For each $x \in u$, $g(x)$ is infinite. Fix a linear ordering of w which is a set and for each $x \in u$, let $f(x)$ be the first element of w such that $\langle x, f(x)\rangle \in r_{g(\alpha)}$. Evidently, $x \doteq f(x)$ for each $x \in u$. Since f maps u into w and since $w \hat{\lesssim} \gamma$ and $\gamma^2 \hat{\lesssim} u$, there is a $y \in w$ such that the set $v = \{x \in u;\ f(x) = y\}$ is infinite. For each $x \in v$ we have $x \doteq f(x) = y$; thus for each $x_1, x_2 \in v$ we have $x_1 \doteq x_2$.

A relation $\doteq$ is called an <u>indiscernibility equivalence</u> iff $\doteq$ is a compact π-equivalence.

It follows by the preceding theorem that the equivalence of infinite nearness of rational numbers defined in Chapter II Section 2 is an indiscernibility equivalence.

<u>Theorem</u>. Let $\{=_n;\ n \in FN\}$ be a sequence of indiscernibility equivalences. Then $\bigcap \{=_n;\ n \in FN\}$ is an indiscernibility equivalence.

<u>Proof</u>. Evidently, $\bigcap \{=_n;\ n \in FN\}$ is an equivalence and a π-class. It remains to prove that it is compact. Let u be an infinite set. There is a descending sequence $\{v_n;\ n \in FN\}$ of subsets of u such that for each n and each $x,y \in v_n$ we have $x =_n y$ and v_n is infinite. The intersection of the sets v_n has at least two elements $x \neq y$ (by the axiom of prolongation). Thus $x,y \in u$ and $x =_n y$ for each n.

An equivalence $=_1$ is <u>finer</u> than $=_2$ (and $=_2$ is <u>coarser</u> than $=_1$) if $x =_1 y$ implies $x =_2 y$ for each x,y.

<u>Theorem.</u> If $=_1$ is finer than $=_2$ and $=_1$ is compact then $=_2$ is also compact.

Section 2

Figures

Throughout this section, $\doteq$ denotes a fixed indiscernibility relation and $\{R_n;\ n \in FN\}$ its generating sequence. We put $o(x,n) = \{y;\ \langle x,y\rangle \in R_n\}$.

To the notion of figure we are led by the following observation.

Two indiscernible classes have the same shape and determine the same figures. Figures are defined as follows:

A class X is a figure iff X contains with each x all y such that $x \doteq y$.

We use sets from the universe of sets both for coding objects and for coding classes of objects. In the former case we shall use the term "point" as synonymous with "set".

We shall not always use all properties of indiscernibility relations and therefore we could study more general topologies as in Cantor's set theory. The author has developed general topology in the alternative set theory; interesting results in such topology were obtained by J. Chudáček. But we shall not develop general topology here before deeper motivations for its study in alternative set theory are exhibited.

Theorem. (1) $\emptyset$ and V are figures. If X, Y are figures, then $X \cup Y$, $X \cap Y$ and $X - Y$ are figures. (2) Let $\varphi(X)$ be a property of classes of the extended universe such that $\varphi(X)$ implies that X is a figure. Then $\bigcup \{X;\ \varphi(X)\}$ and $\bigcap \{X;\ \varphi(X)\}$ are figures.

We define the monad of a point x as follows:
$Mon(x) = \{y;\ y \doteq x\}$.

Evidently, Mon(x) is a figure for each x. A class X is a figure iff the monad of each element of X is a subclass of X.

The <u>figure of</u> X is defined for each X as follows:
$Fig(X) = \{ y;\ (\exists x \in X)(x \doteq y)\}$.

<u>Theorem</u>. For each X, Y, x we have the following:

(1) $Fig(X)$ is a figure.

(2) $X \subseteq Y$ implies $Fig(X) \subseteq Fig(Y)$.

(3) If $X \subseteq Y$ and if Y is a figure then $Fig(X) \subseteq Y$.

(4) $Fig(X \cup Y) = Fig(X) \cup Fig(Y)$.

(5) $Fig(X) \cap Fig(Y) = \emptyset$ iff $Fig(X) \cap Y = \emptyset$.

(6) $Mon(x) = Fig(\{x\})$.

<u>Theorem</u>. If X is set-theoretically definable then $Fig(X)$ is a π-class.

<u>Proof</u>. Put $X_n = \bigcup \{ o(x,n);\ x \in X\}$. Each X_n is set-theoretically definable, $X_{n+1} \subseteq X_n$ and $Fig(X) \subseteq \bigcap \{X_n;\ n \in FN\}$. We prove the converse inclusion. Let z be a point such that $z \in X_n$ for each n. Thus for each n there is an $x_n \in X_n$ such that $z \in o(x_n,n)$. Let $u \subseteq X$ be such that $x_n \in u$ for each n and let $\{ r_\alpha;\ \alpha \leq \tau\}$ be a prolongation of $\{R_n;\ n \in FN\}$ on u. There is an $\alpha \notin FN$ and an $x \in u$ such that $\langle x,z\rangle \in r_\alpha$; consequently, $x \in X$ and $z \in o(x,n)$ for each n, which implies $x \doteq z$ and hence $z \in Fig(X)$.

In the sequal we shall introduce various natural notions; it seems to be superfluous to give an explicit motivation for each of them.

X, Y are <u>separable</u> (notation: $Sep(X,Y)$) iff there is a set-theoretically definable class Z such that $Fig(X) \subseteq Z$ and $Fig(Y) \cap Z = \emptyset$.

Evidently, X and Y are separable iff $Fig(X)$ and $Fig(Y)$ are.

<u>Theorem</u>. Let X and Y be two figures and π-classes. Then X, Y are separable iff they are disjoint.

<u>Proof</u>. Assume $X \cap Y = \emptyset$. Let $X = \bigcap \{X_n;\ n \in FN\}$, $Y = \bigcap \{Y_n;\ n \in FN\}$ where all X_n and Y_n are set-theoretically definable and $X_{n+1} \subseteq X_n$, $Y_{n+1} \subseteq Y_n$ for each n. If $X_n \cap Y_n$ were non-empty for each n then we would have $\bigcap \{X_n \cap Y_n;\ n \in FN\} \neq \emptyset$, thus $X \cap Y \neq \emptyset$. Hence $X_n \cap Y_n = \emptyset$ for some n and X, Y are separable. The converse implication is obvious.

The <u>closure</u> of a class X is denoted by $\overline{X}$ and defined as follows:
$\overline{X} = \{x;\ \neg Sep(\{x\}, X)\}$.

Theorem. For each X, Y we have the following:

(1) $\overline{X}$ is a figure.

(2) $Fig(X) = Fig(Y)$ implies $\overline{X} = \overline{Y}$.

(3) $X \subseteq \overline{X}$.

(4) $X \subseteq Y$ implies $\overline{X} \subseteq \overline{Y}$.

(5) $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$.

Theorem. $x \in \overline{X}$ iff $o(x,n) \cap X \neq \emptyset$ for each n.

Proof. First, assume $x \in \overline{X}$ and let n be such that $o(x,n) \cap X = \emptyset$. We prove $o(x,n+1) \cap Fig(X) = \emptyset$. Assume $z \in o(x,n+1) \cap Fig(X)$. Then there is a $y \in X$ such that $y \doteq z$ and thus $\langle y,z \rangle \in R_{n+1}$. But then $\langle x,y \rangle \in R_n$ and $y \in o(x,n) \cap X$, which is a contradiction.

Conversely, let $o(x,n) \cap X \neq \emptyset$ for each n. Assume $x \notin \overline{X}$; then there is a set-theoretically definable class Z such that $Mon(x) \subseteq Z$ and $Z \cap Fig(X) = \emptyset$. But $o(x,n) \cap (V - Z) \neq \emptyset$ for each n, thus there is a $y \in V - Z$ such that $y \doteq x$, and hence $Mon(x) \cap (V - Z) \neq \emptyset$, which is a contradiction.

Theorem. For each X, $\overline{\overline{X}} = \overline{X}$.

Proof. Obviously, $\overline{X} \subseteq \overline{\overline{X}}$. Let $z \in \overline{\overline{X}}$. Take an n; there is a $y \in o(z,n+1) \cap \overline{X}$. Since $y \in \overline{X}$, there is an $x \in X$ such that $x \in o(y,n+1) \cap X$. This implies $x \in o(z,n)$, hence $o(z,n) \cap X \neq \emptyset$. We have proved $z \in \overline{X}$.

Note that the closure operation is here derived from the indiscernibility relation in contradistinction to classical topology where the closure operation is taken as the basic operation.

A class Y is dense in X if $Y \subseteq X \subseteq \overline{Y}$. Obviously, if Y is dense in X then $\overline{Y} = \overline{X}$.

Theorem. Let $\{u_n;\ n \in FN\}$ be a sequence of classes such that, for each n, u_n is a maximal R_n-net on X. Then $\bigcup\{u_n;\ n \in FN\}$ is dense in X.

Proof. Obviously, $\bigcup\{u_n;\ n \in FN\} \subseteq X$. Let $x \in X$. Then for each n there is a $y_n \in u_n$ such that $\langle x,y_n \rangle \in R_n$. This means that $o(x,n) \cap \bigcup\{u_n;\ n \in FN\} \neq \emptyset$ for each n and hence x belongs to the closure of $\bigcup\{u_n;\ n \in FN\}$.

Corollary. Each X has a dense subclass Y which is at most countable.

Theorem. Let X be at most countable. Then there is a set u such that X is dense in u.

Proof. Let $\{u_n; n \in FN\}$ be a sequence of maximal R_n-nets on X. Put $Y = \bigcup\{u_n; n \in FN\}$. Since Y is dense in X, we have $\overline{X} = \overline{Y}$. Let v be a set such that $X \subseteq v$. Put $v_n = \{y \in v; (\exists x \in u_n)(\langle x,y \rangle \in R_n)\}$. We have $X \subseteq v_n \subseteq v$. Thus there is a set u such that $X \subseteq u \subseteq \bigcap\{v_n; n \in FN\}$. For each n, u_n is a maximal R_n-net on u. Hence Y is dense in u and $\overline{u} = \overline{Y} = \overline{X}$.

Theorem. Let X be a figure. Then the following are equivalent:

(1) X is the figure of a set u.

(2) X is a π-class.

(3) X is revealed.

(4) $X = \overline{X}$.

Proof. (1) => (2) since the figure of each set-theoretically definable class is a π-class.
(2) => (3) is trivial since each π-class is revealed.
(3) => (4). Let X be revealed and assume $x \notin X$. Then X and Mon(x) are disjoint classes and Mon(x) is a π-class. Let $Mon(x) = \bigcap\{Z_n; n \in FN\}$ where the $\{Z_n; n \in FN\}$ is a descending sequence of set-theoretically definable classes. If we had $Z_n \cap X \neq \emptyset$ for each n, then $\bigcap\{Z_n; n \in FN\} \cap X$ would be also non-empty. Thus there is an n such that $Z_n \cap X = \emptyset$. Thus $\{x\}$, X are separated and $x \notin \overline{X}$.
(4) => (1). Let $X = \overline{X}$. Let Y be an at most countable dense subclass of X. By the preceding theorem, there is a set u such that Y is dense in u and thus $\overline{Y} = \overline{u}$ and $\overline{u} = X$. Since Fig(u) is a π-class, we have $\overline{Fig(u)} = Fig(u)$ (by the implication (2) => (4), which has been already proved). Thus $\overline{u} = \overline{Fig(u)} = Fig(u)$ and hence $X = Fig(u)$.

A figure X is <u>closed</u> if it has one (and consequently all) of the properties (1) - (4) of the preceding theorem.

In particular, $\emptyset$, V are closed; if X, Y are closed then X, Y are separated iff X, Y are disjoint. By the preceding theorem, the class of all closed figures is codable.

Theorem. Let $\varphi(X)$ be a property of classes of the extended universe such that $\varphi(X)$ implies $X = \overline{X}$. Then the class $\bigcap\{X; \varphi(X)\}$ is a closed figure.

Proof. Put $Y = \bigcap \{X;\ \varphi(X)\}$. By the preceding theorem, Y is the intersection of a system of figures and hence Y is a figure. For each X satisfying φ we have $Y \subseteq X$ and $\overline{Y} \subseteq X$; thus $\overline{Y} \subseteq \bigcap \{X;\ \varphi(X)\} = Y$, thus $\overline{Y} = Y$.

Theorem. There is an at most countable class Z such that for each closed figure X we have $X = \bigcap \{Fig(u);\ u \in Z\ \&\ X \subseteq Fig(u)\}$.

Proof. Let Y be a dense subclass of V which is at most countable. For each $y \in Y$, let $u(y,n)$ be a set such that $Fig(u(y,n)) = Fig(V - o(y,n))$. (There is such a set). Put $Z = \{u(y,n);\ y \in Y\ \&\ n \in FN\}$. Let X be a closed figure. It remains to prove that for $x \notin X$ there are $y \in Y$ and $n \in FN$ such that $X \subseteq Fig(u(y,n))$ and $x \notin Fig(u(y,n))$. There is an n such that $o(x,n) \cap X = \emptyset$. There is a $y \in Y$ such that $\langle x,y \rangle \in R_{n+2}$. Thus $x \in o(y,n+2)$; now, $o(y,n+1) \subseteq o(x,n)$ and consequently, $X \subseteq Fig(u(y,n+1))$. We prove $x \notin Fig(u(y,n+1))$. Assume the contrary. Then there is a $z \notin o(y,n+1)$ such that $x \doteq z$. But $\langle y,x \rangle \in R_{n+2}$ and $\langle x,z \rangle \in R_{n+2}$, hence $\langle y,z \rangle \in R_{n+1}$, a contradiction. This completes the proof.

Theorem. For each class Z there is a subclass Z' of Z such that Z' is at most countable and $\bigcap \{Fig(u);\ u \in Z\} = \bigcap \{Fig(u);\ u \in Z'\}$.

Proof. Let Z_1 be the class guaranteed by the preceding theorem. Put $Z_2 = \{u \in Z_1;\ (\exists v \in Z)(Fig(v) \subseteq Fig(u))\}$. For each $u \in Z_2$, u' denotes a fixed element of Z such that $Fig(u') \subseteq Fig(u)$. Put $Z' = \{u';\ u \in Z_2\}$. It remains to prove that $x \in \bigcap \{Fig(u);\ u \in Z'\}$ implies $x \in \bigcap \{Fig(u);\ u \in Z\}$. Let x be such that $x \notin \bigcap \{Fig(u);\ u \in Z\}$. Then there is a $u \in Z$ such that $x \notin Fig(u)$. Thus there is a $v \in Z_1$ such that $x \notin Fig(v)$ and $Fig(u) \subseteq Fig(v)$. Since $v \in Z_2$, we have $Fig(v') \subseteq Fig(v)$; consequently, $x \notin Fig(v')$ and $x \notin \bigcap \{Fig(u);\ u \in Z'\}$. This completes the proof.

A class Z is <u>centered</u> iff $\emptyset \neq Z$ and, for each finite non-empty subset w of Z, $\bigcap \{Fig(u);\ u \in w\} \neq \emptyset$.

Theorem. If Z is centered then $\bigcap \{Fig(u);\ u \in Z\} \neq \emptyset$.

Proof. Let Z' be a subclass of Z which is at most countable such that $\bigcap \{Fig(u);\ u \in Z\} = \bigcap \{Fig(u);\ u \in Z'\}$. Since $Fig(u)$ is a π-class for each u and since $\{Fig(u);\ u \in Z'\}$ is a class which is at most countable such that the intersection of finitely many arbitrary elements of this class is non-empty, the intersection

$\bigcap\{\mathrm{Fig}(u);\ u \in Z'\}$ must be also non-empty.

The power-equivalence $\doteq_p$ is defined as follows:
$u \doteq_p v \equiv \mathrm{Fig}(u) = \mathrm{Fig}(v)$.

<u>Theorem</u>. $\doteq_p$ is a π-equivalence.

<u>Proof</u>. Put $\langle u,v\rangle \in R_n^p$ iff $(\forall x \in u)(\exists y \in v)(\langle xy\rangle \in R_n)$ & & $(\forall y \in v)(\exists x \in u)(\langle x,y\rangle \in R_n)$. For each n, R_n^p is set-theoretically definable and $R_{n+1}^p \subseteq R_n^p$. One easily verifies that $\mathrm{Fig}(u) = \mathrm{Fig}(v)$ iff $\langle u,v\rangle \in R_n^p$ for each n. Thus $\doteq_p$ equals to the class $\bigcap\{R_n^p;\ n \in FN\}$.

<u>Theorem</u>. $\doteq_p$ is compact.

<u>Proof</u>. By a theorem in section 1, it suffices to find for each $\gamma \notin FN$ a set w such that $w \mathrel{\hat{\precsim}} \gamma$ and such that for each u there is a $v \in w$ such that $\mathrm{Fig}(u) = \mathrm{Fig}(v)$. Let $\gamma \notin FN$. Take a $\gamma_0 \notin FN$ such that $2^{\gamma_0} \mathrel{\hat{\precsim}} \gamma$. Let w_0 be a set having at most γ_0 elements and such that $(\forall x)(\exists y \in w_0)(x \doteq y)$. Put $w = P(w_0)$. Then w has at most γ elements. Let u be a set and let $\{r_\alpha;\ \alpha \leq \tau\}$ be a prolongation of $\{R_n;\ n \in FN\}$ on $u \cup w_0$. Let f be a function which is a set and assigns to each $x \in u$ an element $f(x) \in w_0$ in such a way that whenever $\langle x,y\rangle \in r_\alpha$ for a $y \in w_0$ and $\alpha \leq \tau$, then $\langle x,f(x)\rangle \in r_\alpha$. Clearly, such an f exists and for each $x \in u$ we have $f(x) \doteq x$. Consequently, putting $v = f''u$ we have $v \subseteq w_0$, i.e. $v \in w$, and $\mathrm{Fig}(v) = \mathrm{Fig}(u)$.

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Section 3

Connectedness

We base our study of connectedness on the notion of a connected set. The following definition is self-explanatory:

A set u is <u>connected</u> (notation: Cntd(u)) iff for each non-empty proper subset v of u there are $x \in v$ and $y \in u - v$ such that $x \doteq y$.

It is useful to introduce the following notion parametrizing the notion of connectedness:

Cntd(u,n) iff for each non-empty proper subset v of u there are $x \in v$ and $y \in u - v$ such that $\langle x,y \rangle \in R_n$.

<u>Theorem</u>. u is connected iff $(\forall n)\mathrm{Cntd}(u,n)$.

<u>Proof</u>. The implication => is obvious. Assume Cntd(u,n) for each n and let v be a proper non-empty subset of u. For each n, let x_n, y_n be such that $x_n \in v$, $y_n \in u - v$ and $\langle x_n, y_n \rangle \in R_n$. By the axiom of prolongation, there are x, y such that $x \in v$, $y \in u - v$ and $(\forall n)(\langle x,y \rangle \in R_n)$, thus $x \doteq y$.

<u>Theorem</u>. If u is a set and if there are two disjoint closed figures X, Y such that $u \subseteq X \cup Y$ and both $X \cap u$ and $Y \cap u$ are non-empty then u is not connected.

<u>Proof</u>. $X \cap u$ and $Y \cap u$ are π-classes; since $Y \cap u = u - (X \cap u)$, $Y \cap u$ is also a σ-class. Thus $Y \cap u$ is set-theoretically definable. Since $Y \cap u$ is also a semiset, $Y \cap u$ is a set. Similarly, $X \cap u$ is a set. Since $\mathrm{Fig}(X \cap u) \subseteq X$ and $\mathrm{Fig}(Y \cap u) \subseteq Y$, we have $\mathrm{Fig}(X \cap u) \cap \mathrm{Fig}(Y \cap u) = \emptyset$. Consequently, u is not connected.

A figure X is <u>connected</u> iff for each $x,y \in X$ there is a connected set $u \subseteq X$ such that $x,y \in u$.

As we shall see below, our definition of connected figures agrees with the classical definition of connectedness for closed figures. Other connected figures correspond to the so-called semi--continua of classical topology.

Theorem. Assume $X = \mathrm{Fig}(u)$. Then X is connected iff u is connected.

Proof. If $\mathrm{Cntd}(u)$ and $x,y \in X$ then obviously $\mathrm{Cntd}(u \cup \{x,y\})$, thus X is connected. Conversely, assume that u is not connected and let v be a proper non-empty subset of u such that $\mathrm{Fig}(v) \cap \mathrm{Fig}(u-v) = \emptyset$. Take $x \in \mathrm{Fig}(v)$ and $y \in \mathrm{Fig}(u-v)$. Let $w \subseteq X$ and $x,y \in w$. Then $w \subseteq \mathrm{Fig}(v) \cup \mathrm{Fig}(u-v)$, $w \cap \mathrm{Fig}(v) \neq \emptyset$ and $w \cap \mathrm{Fig}(u-v) \neq \emptyset$. Hence, by the preceding theorem, w is not connected. Consequently, X is not connected.

Theorem. If $\mathrm{Fig}(v) \subseteq \mathrm{Fig}(u)$ then there is a $w \subseteq u$ such that $\mathrm{Fig}(v) = \mathrm{Fig}(w)$.

Proof. Let $\{ r_\alpha ; \alpha \leq \tau \}$ be a prolongation of $\{ R_n ; n \in FN \}$ on $u \cup v$. Let f be defined on v such that for each $x \in v$ we have $f(x) \in u$ and whenever $\langle x,y \rangle \in r_\alpha$ for some $y \in u$ and $\alpha \leq \tau$ then $\langle x,f(x) \rangle \in r_\alpha$. (The existence is evident.) Since $\mathrm{Fig}(v) \subseteq \mathrm{Fig}(u)$, we have $x \doteq f(x)$ for each $x \in v$. Putting $w = f''v$ we have $w \subseteq u$ and $\mathrm{Fig}(w) = \mathrm{Fig}(v)$.

Theorem. If $\{ X_n ; n \in FN \}$ is a decreasing sequence of closed connected figures then $\bigcap \{ X_n ; n \in FN \}$ is a closed connected figure.

Proof. It suffices to prove that $\bigcap \{ X_n ; n \in FN \}$ is connected. Let $x,y \in \bigcap \{ X_n ; n \in FN \}$. By the preceding theorem, there is a decreasing sequence $\{ u_n ; n \in FN \}$ of connected sets such that $\mathrm{Fig}(u_n) = X_n$ and $x,y \in u_n$ for each n. In particular, we have $\mathrm{Cntd}(u_n, n)$ for each n. By the axiom of prolongation, there is a set u such that $x,y \in u$ and for each n we have $u \subseteq u_n$ and $\mathrm{Cntd}(u,n)$, i.e. u is connected. Since $u \subseteq X_n$ for each n, we have $u \subseteq \bigcap \{ X_n ; n \in FN \}$.

Theorem. For each figure X, the following are equivalent:

(1) X is set-theoretically definable;

(2) X and $V - X$ are closed figures.

Proof. If X is set-theoretically definable then so is $V-X$, thus both X and $V-X$ are π-classes (and we know that a figure is closed iff it is a π-class). Conversely, if both X and $V-X$ are π-classes then X is both a π-class and a σ-class and hence X is set-theoretically definable.

A figure X is called <u>clopen</u> if it has one of the properties (1), (2) from the preceding theorem (then X has both (1) and (2)).

<u>Theorem</u>. The class of all clopen figures is codable and at most countable.

<u>Proof</u>. Codability follows from the codability of all set-theoretically definable classes. In Section 2 we proved that there is a $Z \precsim FN$ such that for each closed figure X there is a $Z' \subseteq Z$ such that $X = \bigcap\{Fig(u);\ u \in Z'\}$. If X is set-theoretically definable then Z' can be taken to be finite. Thus if we associate with each clopen figure X a finite set $w \subseteq Z$ such that $X = \bigcap\{Fig(u);\ u \in w\}$, then we have constructed a one-one mapping of all clopen figures into the countable class P(Z) of all (finite) subsets of Z.

The equivalence of connectedness $\doteq_C$ is defined as follows: $x \doteq_C y$ iff there is a connected set u such that $x,y \in u$.

Obviously, $x \doteq y$ implies $x \doteq_C y$, i.e. $\doteq_C$ is coarser than $\doteq$. Consequently, $\doteq_C$ is compact.

Put $R^C_n = \{\langle x,y\rangle;\ (\exists u)(Cntd(u,n)\ \&\ x,y \in u)\}$.

For each n, R^C_n is a set-theoretically definable equivalence and $R^C_{n+1} \subseteq R^C_n$. Furthermore, $\doteq_C$ is a subrelation of R^C_n; thus for each x, the class $\{y;\ \langle x,y\rangle \in R^C_n\}$ is a figure. Since it is set-theoretically definable, it is a clopen figure.

<u>Theorem</u>. $\doteq_C$ coincides with $\bigcap\{R^C_n;\ n \in FN\}$.

<u>Proof</u>. If $\langle x,y\rangle \in R^C_n$ for each n then there is a sequence $\{u_n;\ n \in FN\}$ such that, $x,y \in u_n$ and $Cntd(u_n,n)$ for each n. By the axiom of prolongation, there is a u such that $x,y \in u$ and $Cntd(u,n)$ for each n. Thus $x \doteq_C y$. The converse inclusion is trivial.

<u>Corollary</u>. $\doteq_C$ is an indiscernibility equivalence.

The monad of a point x w.r.t. $\doteq_C$ is called the <u>component</u> of x.

<u>Theorem</u>. $x \doteq_C y$ iff there is no clopen figure X containing x but not y.

<u>Proof</u>. If there is a clopen figure X with $x \in X$ and $y \notin X$ then, by a theorem above, there is no connected set u such that $x,y \in u$. On the other hand, if there is no such figure then $\langle x,y\rangle \in R^C_n$ for

each n and hence $x \doteq_C y$ by the preceding theorem.

<u>Theorem</u>. The following are equivalent:

(1) The equivalences $\doteq$ and $\doteq_C$ coincide.

(2) $\doteq$ has a generating sequence $\{S_n;\ n \in FN\}$ such that S_n is an equivalence for each n.

<u>Proof</u>. Assume (1). Then $\{R_n^C;\ n \in FN\}$ is a generating sequence of $\doteq$ having the desired property.

Conversely, assume (2) and let x, y be such that not $x \doteq y$. Then there is an n such that $\langle x,y\rangle \notin S_n$. The class $\{z;\ \langle x,z\rangle \in S_n\}$ is a set-theoretically definable figure containing x but not y. By the preceding theorem, we have not $x \doteq_C y$. Thus $\doteq$ and $\doteq_C$ coincide.

The equivalence $\doteq$ is said to be totally disconnected if it satisfies one of the conditions (1), (2) (and hence both of them).

<u>Theorem</u>. $\doteq_C$ is totally disconnected.

<u>Proof</u>. $\{R_n^C,\ n \in FN\}$ is a generating sequence of $\doteq_C$ and each R_n^C is an equivalence.

<u>Theorem</u>. The following are equivalent:

(1) $\doteq$ is set-theoretically definable.

(2) Mon(x) is a clopen figure for each x.

(3) The class $V/\doteq$ is finite.

<u>Proof</u>. (1) => (2) is trivial. We prove (2) => (3). Assume (2). Since $\{Mon(x);\ x \in V\}$ is a class of clopen figures, it is at most countable. Since its union is V it is finite.

We prove (3) => (1). Assume (3). For each x, V - Mon(x) is a closed figure (as a union of finitely many closed figures). Since Mon(x) is also closed, Mon(x) is clopen. Thus V is decomposed into finitely many set-theoretically definable classes; it follows that the corresponding equivalence is set-theoretically definable.

The equivalence $\doteq$ is called <u>discrete</u> if it has one of the properties of the preceding theorem (and hence all of them).

Evidently, if $\doteq$ is discrete then $\doteq$ is totally disconnected.

Chapter IV

Motion

Motion is one phenomenon of the continuum that has been traditionally studied by mathematical means.

In the alternative set theory, we shall deal with motion on the basis of the hypothesis concerning phenomena of the continuum formulated in the introduction to Chapter III. This hypothesis is specified for the case of motion as follows:

Motion is a phenomenon which we perceive when we are presented with a sequence of states in which each state differs indistinguishably from the preceding state in time and substance.

Thus, for example, if we watch a movie we are presented with a rapid series of photographs. We are unable to perceive the differences between successive photographs since these differences lie beyond the horizon of our ability to distinguish. The series of photographs is then perceived as motion.

Changes realized in the passage from one state to the next state are called infinitesimal. Since we work in the extended universe we shall try to model states as sets or classes.

The description of the phenomenon of motion on the basis of infinitesimal changes raises the question of the relationship between infinitesimal and global properties of the motion. Thus, for example, the growth of a plant from a seed is a motion. The problem is to determine the resulting shape of the plant from the sequence of infinitesimal changes of shape, or, conversely, to derive this sequence from the global growth.

This example shows that we understand motion here in a rather general manner and do not restrict ourselves to mechanical motions.

Section 1

Motions of points

Throughout this chapter, $\doteq$ denotes an indiscernibility equivalence and $\{R_n;\ n \in FN\}$ denotes a generating sequence of $\doteq$.

The motion of a point is the simplest kind of motion. A point moves by changing permanently, in imperceptible jumps, its location. With each jump it must not disappear and suddenly reappear in a distant place; thus after each jump it must find itself in a place indiscernible from the previous place. This leads to the following definition.

A function d is a <u>motion of a point</u> in a time ϑ, where $\vartheta \in N$, if (1) $dom(d) = \vartheta + 1$ and (2) for each $\alpha < \vartheta$, $d(\alpha) \doteq d(\alpha+1)$.

Thus, the time of the motion d is the number of states visited by the point when moving from $d(0)$ to $d(\vartheta)$.

If d is a motion of a point then $rng(d)$ is called the <u>trace</u> of d

<u>Theorem</u>. The trace of a motion of a point is a connected set.

<u>Proof</u>. Let v be a proper non-empty subset of $rng(d)$. Assume $d(0) \in v$. Let γ be the maximal α such that for each $\beta \leq \alpha$ we have $d(\beta) \in v$. We have $\gamma < \vartheta$, $d(\gamma) \in v$, $d(\gamma+1) \in rng(d) - v$ and $d(\gamma) \doteq d(\gamma+1)$. Similarly for $d(0) \in rng(d) - v$.

<u>Theorem</u>. For each non-empty connected set u there is a motion d of a point such that u is the trace of d.

<u>Proof</u>. Let $\{r_\alpha;\ \alpha \leq \tau\}$ be a prolongation of $\{R_n;\ n \in FN\}$ on u. We claim that for each n there is a function f and a $\vartheta_n \in N$ such that $dom(f) = \vartheta_n + 1$, $rng(f) = u$ and for each $\alpha < \vartheta_n$ we have $\langle f(\alpha),\ f(\alpha+1)\rangle \in r_n$. By the axiom of prolongation, there is a infinite $\gamma < \tau$ and a function f such that $dom(f) = \vartheta_\gamma \in N$, $rng(f) = u$, and $\langle f(\alpha),\ f(\alpha+1)\rangle \in r_\gamma$ for each $\alpha < \vartheta_\gamma$, thus $f(\alpha) \doteq f(\alpha+1)$ for each $\alpha < \vartheta_\gamma$. Thus f is the desired motion.

Indeed, take an n. Put $Y = \{f;\ dom(f) \in N\ \&\ rng(f) \subseteq u\ \&$ $\&\ (\forall \alpha)(\alpha + 1 \in dom(f) \Rightarrow \langle f(\alpha), f(\alpha + 1)\rangle \in r_n)\}$. The class is set-theoretically definable. Thus there is a $g \in Y$ such that $f \in Y$ implies $rng(f) \mathrel{\hat{\lesssim}} rng(g)$ ($rng(g)$ has the largest possible number of elements). We show $rng(g) \mathrel{\hat{\approx}} u$, which implies $rng(g) = u$ since $rng(g) \subseteq u$. Assume that $rng(g)$ is a proper subset of u. Since $rng(g) \neq \emptyset$, there is a $\beta \in dom(g)$ and $y \in u - rng(g)$ such that $g(\beta) \doteq y$. In particular, $\langle g(\beta), y\rangle \in r_n$. Put $dom(g) = \vartheta + 1$. We define a function f on $\vartheta + (\vartheta - \beta) + 2$ as follows:

$$f(\alpha) = \begin{cases} g(\alpha) & \text{for } \alpha \leq \vartheta, \\ g(\alpha - 2(\alpha - \vartheta)) & \text{for } \vartheta < \alpha \leq 2\vartheta - \beta, \\ y & \text{for } \alpha = \vartheta + (\vartheta - \beta) + 1. \end{cases}$$

Evidently, $f \in Y$ and $rng(f) = rng(g) \cup \{y\}$, which is a contradiction.

In the rest of this section, d denotes a motion of a point in a time ϑ, $\vartheta \in N$. For $\alpha, \beta \leq \vartheta$ put $\alpha \mathrel{\underset{d}{\doteq}} \beta$ iff for each γ satisfying $Min\{\alpha, \beta\} \leq \gamma \leq Max\{\alpha, \beta\}$ we have $d(\alpha) \doteq d(\gamma)$.

Obviously, $\alpha \mathrel{\underset{d}{\doteq}} \beta$ implies $d(\alpha) \doteq d(\beta)$. Thus $\underset{d}{\doteq}$ is finer than the indiscernibility equivalence $\{\langle\alpha, \beta\rangle;\ d(\alpha) \doteq d(\beta)\}$.

<u>Theorem</u>. $\underset{d}{\doteq}$ is a π-equivalence.

<u>Proof</u>. For each $\alpha, \beta \leq \vartheta$, put
$\langle\alpha, \beta\rangle \in S_n \equiv (\forall \gamma$ between $Min\{\alpha, \beta\}, Max\{\alpha, \beta\})(\langle d(\alpha), d(\gamma)\rangle \in R_n)$
(γ is <u>between</u> α_1, α_2 if $\alpha_1 \leq \gamma \leq \alpha_2$). Evidently, each S_n is set-theoretically definable and it is easy to verify that $\alpha \mathrel{\underset{d}{\doteq}} \beta$ iff $\langle\alpha, \beta\rangle \in S_n$ for each n.

The motion d is called compact if $\underset{d}{\doteq}$ is a compact equivalence (a fortiori, $\underset{d}{\doteq}$ is an indiscernibility equivalence).

<u>Theorem</u>. The motion d is compact iff for each x and each infinite set $u \subseteq \{\alpha;\ d(\alpha) \doteq x\}$ there are $\alpha, \beta \in u$ such that $\alpha \neq \beta$ and $\alpha \mathrel{\underset{d}{\doteq}} \beta$.

<u>Proof</u>. The implication $\Rightarrow$ is obvious. Conversely, assume that d is not compact. Let $v \subseteq \vartheta + 1$ be an infinite set such that, for each $\alpha, \beta \in v$, $\alpha \mathrel{\underset{d}{\doteq}} \beta$ implies $\alpha = \beta$. Since $\doteq$ is compact, there is a point x and an infinite $u \subseteq v$ such that $d''u \subseteq Mon(x)$. Thus $u \subseteq \{\alpha;\ d(\alpha) \doteq x\}$ and the condition of our theorem is satisfied.

The cardinality of the class $\{\alpha;\ d(\alpha) \doteq x\}/\underset{d}{\doteq}$, i.e. the number of equivalence classes of $\underset{d}{\doteq}$ on $\{\alpha;\ d(\alpha) \doteq x\}$, is called the <u>order</u> of the point x. Thus the order of x tells how many times the motion d enters the monad of x.

The last theorem has the following

<u>Corollary</u>. If the order of each point is at most countable then d is compact.

The motion d <u>oscillates between points</u> x, y (<u>between sets</u> u, v) if $\mathrm{Mon}(x) \cap \mathrm{Mon}(y) = \emptyset$ $(\mathrm{Fig}(u) \cap \mathrm{Fig}(v) = \emptyset)$ and there are sequences $\{\alpha_n;\ n \in FN\}$, $\{\beta_n;\ n \in FN\}$ such that $\alpha_n < \beta_n < \alpha_{n+1} < \beta_{n+1}$ for each n and $d(\alpha_n) \doteq x$, $d(\beta_n) \doteq y$ $(d(\alpha_n) \in u,\ d(\beta_n) \in v)$.

The following theorem is an immediate consequence of the axiom of prolongation.

<u>Theorem</u>. d oscillates between points x, y (between sets u, v) iff $\mathrm{Mon}(x) \cap \mathrm{Mon}(y) = \emptyset$ $(\mathrm{Fig}(u) \cap \mathrm{Fig}(v) = \emptyset)$ and there are sequences $a = \{\alpha_\gamma;\ \gamma \leq \delta\}$, $b = \{\beta_\gamma;\ \gamma \leq \delta\}$ (i.e. a, b are sets) such that $\delta \notin FN$ and for each $\gamma < \delta$ we have $\alpha_\gamma < \beta_\gamma < \alpha_{\gamma+1} < \beta_{\gamma+1}$ and $d(\alpha_\gamma) \doteq x$, $d(\beta_\gamma) \doteq y$ $(d(\alpha_\gamma) \in u,\ d(\beta_\gamma) \in v)$.

<u>Theorem</u>. The following are equivalent:

(1) The motion d is not compact.

(2) There are points x, y such that d oscillates between x, y.

(3) There are sets u, v such that d oscillates between u, v.

<u>Proof</u>. (1) $\Rightarrow$ (2). By a theorem above, there is a point x and an infinite set $\{\alpha_\gamma;\ \gamma \leq \delta\}$ such that for each $\gamma < \delta$ we have $\alpha_\gamma < \alpha_{\gamma+1}$, $d(\alpha_\gamma) \doteq x$ but not $\alpha_\gamma \doteq_d \alpha_{\gamma+1}$. For each γ, let $\bar{\gamma}$ be the least natural number such that there is a β such that $\alpha_\gamma < \beta < \alpha_{\gamma+1}$ and $\langle d(\alpha_\gamma), d(\beta) \rangle \notin R_{\bar{\gamma}}$. Obviously, $\bar{\gamma} \in FN$. Let β_γ be the least β such that $\alpha_\gamma < \beta < \alpha_{\gamma+1}$ and $\langle d(\alpha_\gamma), d(\beta) \rangle \notin R_{\bar{\gamma}}$. The sequence $\{\beta_\gamma;\ \gamma \leq \delta\}$ is a set and there is a point y and an infinite set $v \subseteq \delta$ such that $\gamma \in v$ implies $d(\beta_\gamma) \doteq y$ for each γ. Since <u>not</u> $d(\alpha_\gamma) \doteq d(\beta_\gamma)$, we have <u>not</u> $x \doteq y$. Thus the sequences $\{\alpha_\gamma;\ \gamma \in v\}$, $\{\beta_\gamma;\ \gamma \in v\}$ witness the oscillation of d between the points x, y.

(2) $\Rightarrow$ (3). Let d oscillate between the points x, y and let the sequences $\{\alpha_\gamma;\ \gamma \leq \delta\}$, $\{\beta_\gamma;\ \gamma \leq \delta\}$ witness this oscillation. There are sets u, v such that $u \subseteq \mathrm{Mon}(x)$, $v \subseteq \mathrm{Mon}(y)$ and for each $\gamma \leq \delta$, $d(\alpha_\gamma) \in u$, $d(\beta_\gamma) \in v$. Then d oscillates between the sets u, v.

(3) => (1). By the preceding theorem, there are u, v such that $\mathrm{Fig}(u) \cap \mathrm{Fig}(v) = \emptyset$ and sequences $\{\alpha_\gamma ; \gamma \le \delta\}$, $\{\beta_\gamma ; \gamma \le \delta\}$ (sets!) such that $\delta \notin FN$ and for each $\gamma \le \delta$ we have $\alpha_\gamma < \beta_\gamma < \alpha_{\gamma+1} < \beta_{\gamma+1} < \vartheta$, $d(\alpha_\gamma) \in u$, $d(\beta_\gamma) \in v$. Thus $\gamma < \delta$ implies <u>not</u> $\alpha_\gamma \doteq_d \alpha_{\gamma+1}$ and consequently, for each $\gamma, \beta \in \delta$, $\alpha_\gamma \doteq_d \alpha_\beta$ implies $\gamma = \beta$. Thus d is not compact.

When we follow a motion d of a point then we have an indiscernibility equivalence $\doteq_T$ on its duration $\vartheta + 1$, satisfying the following natural condition:

$$(\alpha < \gamma < \beta \;\&\; \alpha \doteq_T \beta) \Rightarrow \alpha \doteq_T \gamma \text{, for all } \alpha, \beta, \gamma.$$

The kind of our evidence of d depends on the relationship between $\doteq_d$ and $\doteq_T$.

For example, if $\doteq_T$ is finer than $\doteq_d$ then our evidence of d makes it possible to determine for each monad of time the corresponding monad of the position of the moving point, i.e. a monad in $\doteq$. Since $\doteq_T$ is compact, this is possible only if d is compact.

On the other hand, if a monad of $\doteq_T$ intersects more than one monad of $\doteq_d$, then in such a monad of time we cannot determine a single monad of position as the corresponding position of the moving point.

Further properties of motions of points depending on metric properties of $\doteq$ and $\doteq_T$ will not be studied here since we have not yet introduced the necessary notions.

Section 2

Pointwise motions of sets

We are now going to formulate a definition of a motion of a set; this definition will be in accord with our concept of motion but will concern only the simple case that the moving set is not endowed with any structure.

A function d is a <u>motion of a set</u> in the time ϑ, where ϑ is a natural number, iff $\mathrm{dom}(d) = \vartheta + 1$ and, for each $\alpha < \vartheta$, $\mathrm{Fig}(d(\alpha)) = \mathrm{Fig}(d(\alpha + 1))$.

Clearly, d is a motion of a set in the time ϑ iff d is a motion of a point in the time ϑ with respect to the power-equivalence $\doteq_P$ of $\doteq$. Since motions of sets are particular motions of points all results of the preceding section apply to motions of sets.

The aim of the present section is to formulate infinitesimal conditions equivalent to the decomposability of a motion of a set into motions of single points. If some points move simultaneously then it is impossible that two distinct points have the same position. This motivates the following definition:

A class T is a <u>sheaf of motions of points</u> in the time ϑ iff we have the following:

(1) Each element of T is a motion of a point in the time ϑ.

(2) If f, g are distinct elements of T and if $\alpha \leq \vartheta$ then $f(\alpha) \neq g(\alpha)$.

We put $T_\alpha = \{f(\alpha);\ f \in T\}$ for each $\alpha \leq \vartheta$ and

$$T_{\alpha,\beta} = \{\langle f(\alpha),\ f(\beta)\rangle;\ f \in T\} \text{ for each } \alpha, \beta \leq \vartheta .$$

Evidently, $T_{\alpha,\beta}$ is a one-one mapping of T_β onto T_α. Moreover, $T_{\alpha,\alpha+1}$ associates with each point an indiscernible point.

Classes X, Y are <u>infinitesimally shifted</u> (notation: $\mathrm{Shftd}(X,Y)$) iff for each z, $\mathrm{Mon}(z) \cap X \approx \mathrm{Mon}(z) \cap Y$ (i.e. there is a one-one mapping of $\mathrm{Mon}(z) \cap X$ to $\mathrm{Mon}(z) \cap Y$).

Evidently, if X, Y are infinitesimally shifted then

$\mathrm{Fig}(X) = \mathrm{Fig}(Y)$. In particular, if T is a sheaf of motions in the time ϑ and if $\alpha < \vartheta$, then T_α, $T_{\alpha+1}$ are infinitesimally shifted.

A motion d of a set in the time ϑ is a *pointwise motion of a set* if, for each $\alpha < \vartheta$, $d(\alpha)$ and $d(\alpha + 1)$ are infinitesimally shifted.

It follows from the above considerations that if a motion d of a set is decomposable into motions of its points then d is a pointwise motion of a set. We shall prove that the converse implication is also true. Moreover, we shall be able to decompose a pointwise motion of a set into motions of points in such a way that our decomposition will have useful additional properties.

A sheaf T of motions of points in the time ϑ is a *strong decomposition* of a pointwise motion d of a set iff we have the following:

(1) For each $\alpha \leq \vartheta$, $T_\alpha = d(\alpha)$.

(2) For each $\alpha, \beta \leq \vartheta$ and each $u \subseteq T_\beta$, $T''_{\alpha,\beta}\, u$ is a set.

(3) For each $u \subseteq T_0$, the function $\{\langle T''_{\alpha,0}\, u, \alpha\rangle ; \alpha \leq \vartheta\}$ associating with each α the set $T''_{\alpha,0} u$ is itself a set.

Condition (1) says that T decomposes d into motions of points. Conditions (2), (3) say that T induces a motion of subsets of the set moving in the motion d.

We shall now formulate the main theorem of this section.

Theorem. Let d be a pointwise motion of a set in the time ϑ. Then there is a sheaf T of motions of points in the time ϑ which is a strong decomposition of d.

We first formulate two simple corollaries.

Corollary. The following are equivalent:

(1) u and v are infinitesimally shifted.

(2) There is a one-one mapping F of u onto v such that $x \doteq F(x)$ for each $x \in u$.

(3) There is a one-one mapping F of u onto v such that $x \doteq F(x)$ for each $x \in u$ and $\mathrm{Set}(X) \equiv \mathrm{Set}(F''X)$ for each $X \subseteq u$.

Corollary. Let u, v be infinite sets. Then there is a one-one mapping F of u onto v such that, for each $X \subseteq u$, X is a set iff $F''X$ is a set.

We shall now prove our main theorem. The proof has been considerably simplified by K. Čuda and J. Mlček. All theorems and notions in the rest of this section are auxiliary to this proof.

In the sequal, c denotes a set which will be specified in the proofs of our auxiliary theorems and $\{r_\alpha;\ \alpha \leq \tau\}$ denotes a prolongation of $\{R_n;\ n \in FN\}$ on c. In contradistinction to the preceding chapter, $o(x,\alpha)$ now denotes the set $\{y \in c;\ \langle x,y \rangle \in r_\alpha\}$. For $u \subseteq c$ and $x \in c$ we put $y = S(x,u)$ iff y is the first element of c (in a fixed linear ordering $\preceq$ of c such that $\preceq$ is a set) such that

$o(x,\alpha) \cap u \neq \emptyset \Rightarrow y \in o(x,\alpha) \cap u$

for each $\alpha \leq \tau$.

Evidently, S is a set. For $u \neq \emptyset$ we have $S(x,u) \in u$, for $x \in u$ we have $S(x,u) = x$ and for $x \in Fig(u)$ we have $S(x,u) \doteq x$.

<u>Theorem</u>. Let u, u′ be infinitesimally shifted and let v, w be disjoint sets whose union is u. Then there are disjoint sets v′, w′ whose union is u′ such that v, v′ are infinitesimally shifted and w, w′ are also infinitesimally shifted.

<u>Proof</u>. Let $u \cup u' \subseteq c$. First assume $u \subseteq u'$. Put
$\bar{v} = \{x;\ x \in u'-u\ \&\ Min\{\alpha;\ o(x,\alpha) \cap v \mathrel{\hat{\prec}} \alpha\} \geq Min\{\alpha;\ o(x,\alpha) \cap w \mathrel{\hat{\prec}} \alpha\}\}$;
furthermore, put $v' = v \cup \bar{v}$, $w' = u' - v'$. We prove that v, v′ are infinitesimally shifted. Since $v \subseteq v'$, it suffices to prove that if $Mon(x) \cap v$ is finite then $Mon(x) \cap v = Mon(x) \cap v'$, thus $Mon(x) \cap \bar{v} = \emptyset$. Assume the contrary; then we may assume $x \in \bar{v}$. Consequently, there is an m such that $o(x,m) \cap v = Mon(x) \cap v$. This implies that
$Min\{\alpha;\ o(x,\alpha) \cap v \mathrel{\hat{\prec}} \alpha\} \in FN$. But then also
$Min\{\alpha;\ o(x,\alpha) \cap w \mathrel{\hat{\prec}} \alpha\} \in FN$, since $x \in \bar{v}$. It follows that $Mon(x) \cap w$ is finite and hence $Mon(x) \cap u$ is also finite. Since u, u′ are infinitesimally shifted and $u \subseteq u'$ we have $Mon(x) \cap u = Mon(x) \cap u'$. Thus $x \in u$, which is a contradiction. One shows similarly $Shftd(w, w')$.

We now prove the theorem in full generality. For $\alpha \leq \tau$ we define sets v_α, w_α and functions f_α, g_α by induction as follows: v_α and w_α is a maximal r_α-net on $v - \bigcup\{v_\beta;\ \beta \in \alpha\}$,
$w - \bigcup\{w_\beta;\ \beta \in \alpha\}$ respectively. We put
$f_\alpha(x) = S(x, u' - (\bigcup\{rng(f_\beta);\ \beta \in \alpha\} \cup \bigcup\{rng(g_\beta);\ \beta \in \alpha\}))$ for each $x \in v_\alpha$,
$g_\alpha(x) = S(x, u' - (\bigcup\{rng(f_\beta);\ \beta \leq \alpha\} \cup \bigcup\{rng(g_\beta);\ \beta \in \alpha\}))$ for each $x \in w_\alpha$.
Let δ be the maximal $\alpha \leq \tau$ such that $\bigcup\{f_\beta \cup g_\beta;\ \beta \leq \alpha\}$ is a

one-one mapping, $\langle x, f_\beta(x)\rangle \in r_\beta$ for each $\beta \le \alpha$ and each $x \in v_\beta$, and $\langle x, g_\beta(x)\rangle \in r_\beta$ for each $\beta \le \alpha$ and each $x \in w_\beta$. Since u, u′ are infinitesimally shifted $x \in v_n$ implies $x \doteq f_n(x)$ and $x \in w_n$ implies $x \doteq g_n(x)$ for each n. Thus $\delta \notin FN$. Put $f = \bigcup\{f_\alpha ; \alpha \le \delta\}$ and $g = \bigcup\{g_\beta ; \beta \le \delta\}$. Then Shftd(dom(f), rng(f)), Shftd(dom(g), rng(g)) and $rng(g) \cap rng(f) = \emptyset$. By the first part of the proof, there are v′, w′ such that $v' \cap w' = \emptyset$, $v' \cup w' = u'$, Shftd(rng(f), v′), Shftd(rng(g), w′). Thus it suffices to prove Shftd(v, dom(f)) and Shftd(w, dom(g)). We prove the first equality; the second is proved analogously. Note that $dom(f) \subseteq v$; thus it suffices to prove that if $Mon(x) \cap dom(f)$ is finite then $Mon(x) \cap v \subseteq Mon(x) \cap dom(f)$. Assume the contrary. Let $Mon(x) \cap dom(f)$ be finite and let $x \in v - dom(f)$. (This can be assumed without loss of generality.) Pick a n_1 such that $Mon(x) \cap dom(f) = o(x,n_1) \cap dom(f)$. Furthermore, pick a n_2 such that $o(x,n_1) \cap dom(f) \subseteq \bigcup\{v_n; n \le n_2\}$. Put $m = Max\{n_1, n_2\} + 1$. But then there is a $y \in v_m$ such that $\langle x,y\rangle \in r_m$, hence $y \in o(x,n_1)$ and $y \in dom(f) - \bigcup\{v_n; n \le n_2\}$, which is a contradiction.

An inspection of this proof shows that we have described a set--theoretically definable function associating with each quadruple $\langle u, v, w, u'\rangle$, satisfying the conditions $u = v \cup w$, $v \cap w = \emptyset$, $u \cup u' \subseteq c$, a decomposition $\langle v', w'\rangle$ of u′ into two disjoint sets. Using axioms concerning the extended universe we have shown that Shftd(u, u′) implies Shftd(v, v′) and Shftd(w, w′). Iterating this construction and specifying appropriately c we get easily the following theorem:

<u>Theorem.</u> Let d be a pointwise motion of a set in the time ϑ. Let $\gamma \le \vartheta$ and $v \subseteq d(\gamma)$. Then there are pointwise motions d_1, d_2 of sets in the time ϑ such that $d_1(\gamma) = v$ and, for each $\alpha \le \vartheta$, the pair $\langle d_1(\alpha), d_2(\alpha)\rangle$ is a decomposition of $d(\alpha)$ into two disjoint sets.

For each α such that $\alpha + 2 \le \tau$ we put
$u^{(\alpha)} = \{x; x \in u \ \& \ o(x,\alpha) \cap u = o(x, \alpha+2) \cap u \ \& \ o(x,\alpha) \hat{\le} \alpha\}$.

Let Shftd (α, u, v) be the conjunction of the following two formulas:

$(\forall x \in u^{(\alpha)})(\ o(x, \alpha + 2) \cap u \hat{\approx} o(x, \alpha + 1) \cap v)$,
$(\forall x \in v^{(\alpha)})(\ o(x, \alpha + 2) \cap v \hat{\approx} o(x, \alpha + 1) \cap u)$.

Evidently, Shftd (α, u, v) can be written as a set-formula.

Theorem. u, v are infinitesimally shifted iff $(\forall n)(\mathrm{Shftd}(n, u, v))$.

Proof. First assume Shftd(u,v) and let $n \in FN$. Let, for example, $x \in u^{(n)}$; thus $o(x,n) \cap u = o(x,n+2) \cap u \hat{\lesssim} n$. For each $y \in o(x,n+2)$, $\mathrm{Mon}(y) \cap u$ is finite; since Shftd(u,v) and $o(x,n+2) \cap u = o(x,n+1) \cap u$, there is a one-one mapping f of $o(x,n+2) \cap u$ onto $o(x,n+1) \cap v$; moreov we can assume that f associates with each point an infinitely near point. This proves $o(x,n+2) \cap u \hat{\approx} o(x,n+1) \cap v$.

Conversely, assume that u, v are not infinitesimally shifted. For example let $x \in u$, let $\mathrm{Mon}(x) \cap u$ have k elements (where k is a finite natural number) and let $\mathrm{Mon}(x) \cap v$ have more than k elements. Then there is a $n > k$ such that $\mathrm{Mon}(x) \cap u = o(x,n) \cap u$; thus $x \in u^{(n)}$. But $\mathrm{Mon}(x) \cap v \subseteq o(x,n+1) \cap v$, thus $o(x,n+2) \cap u \hat{\prec} o(x,n+1) \cap v$, and consequently we have not Shftd(n, u, v).

In the rest of the present section, d denotes a fixed pointwise motion of a set in the time ϑ. Furthermore a denotes the set of all functions f such that $\mathrm{dom}(f) = \vartheta + 1$ and $f(\alpha) \subseteq d(\alpha)$ for each $\alpha \leq \vartheta$ and A is the class of all $f \in a$ such that f is a pointwise motion of a set in the time ϑ. For $X \subseteq a$ we put

$$\nu(X) = \{\gamma;\ (\forall \alpha \leq \vartheta)(\forall \beta \leq \gamma)(\forall f \in X)(\mathrm{Shftd}\ (\beta,\ f(\alpha),\ f(\alpha+1)))\}.$$

Evidently, for each $X \subseteq a$ we have $X \subseteq A$ iff $FN \subseteq \nu(X)$. Moreover, if X is a countable subclass of a then $X \subseteq A$ iff FN is a proper subclass of $\nu(X)$.

A class $B \subseteq a$ is a called a (Boolean) algebra on d if we have the following:

(1) if $f \in B$ and g is the function on $\vartheta + 1$ such that $g(\alpha) = d(\alpha) - f(\alpha)$ for each $\alpha \leq \vartheta$ then $g \in B$;

(2) if $f_1, f_2 \in B$ and g is the function on $\vartheta + 1$ such that $g(\alpha) = f_1(\alpha) \cup f_2(\alpha)$ for each $\alpha \leq \vartheta$, then $g \in B$.

For each $X \subseteq a$, B(X) is the intersection of all algebras B on d such that $X \subseteq B$. Evidently, B(X) is an algebra, and if X is finite (countable) then B(X) is also finite (countable). Furthermore, it is evident that B is a countable algebra iff there is an ascending sequence $\{B_n;\ n \in FN\}$ of finite algebras whose union is B.

A finite set p is a partition iff $p \subseteq a$ and

(1) $(\forall f_1, f_2 \in p)(\forall \alpha \leq \vartheta)(f_1 \neq f_2 \Rightarrow f_1(\alpha) \cap f_2(\alpha) = \emptyset)$,

(2) $(\forall \alpha \leq \vartheta)(d(\alpha) = \bigcup \{f(\alpha);\ f \in p\})$.

If p is a partition then
$B(p) = \{f;\ (\exists x \subseteq p)(\forall \alpha \leq \vartheta)(f(\alpha) = \bigcup \{g(\alpha),\ g \in x\})\}$. Thus if $p \subseteq A$ then $B(p) \subseteq A$. Furthermore, B is a finite algebra iff there is a partition p such that $B = B(p)$.

Theorem. Let $B \subseteq A$ be an at most countable algebra, let $\gamma \leq \vartheta$ and $v \subseteq d(\gamma)$. Then there is an at most countable algebra B' such that $B \subseteq B' \subseteq A$ and $v = f(\gamma)$ for some $f \in B'$.

Proof. First assume that B is finite. Let p be a partition such that $B = B(p)$. Evidently, $p \subseteq A$. By our second auxiliary theorem, there is a partition $p' \subseteq A$ such that
$\{f(\gamma);\ f \in p'\} = \{f(\gamma) \cap v;\ f \in p\} \cup \{f(\gamma) - v;\ f \in p\}$.
It suffices to put $B' = B(p)$.

Now assume that B is countable and let $\{B_n;\ n \in FN\}$ be an ascending sequence of algebras whose union is B. Let $\{B'_n;\ n \in FN\}$ be the sequence of corresponding algebras constructed according to the first part of the proof. Choose a $\delta \notin FN$ such that $\delta \in \nu(B_n) \cap \nu(B'_n)$ for each n. By the axiom of prolongation, both sequences have prolongation $\{B_\alpha;\ \alpha \in \alpha_0\}$, $\{B'_\alpha;\ \alpha \in \alpha_0\}$ (sets!) such that, for each $\alpha \in \alpha_0$, $\delta \in \nu(B_\alpha) \cap \nu(B'_\alpha)$, i.e. $B_\alpha \subseteq A$, $B'_\alpha \subseteq A$; moreover, $B_\alpha \subseteq B_{\alpha+1}$, $B_\alpha \subseteq B'_\alpha$, and there is exactly one $f_\alpha \in B'_\alpha$ such that $f_\alpha(\gamma) = v$. Choose an $\alpha \in \alpha_0$, $\alpha \notin FN$ and put $B' = B(B \cup \{f_\alpha\})$. Evidently, $B \subseteq B' \subseteq B'_\alpha \subseteq A$, B' is countable and contains f_α; hence B' is the desired algebra.

Theorem. There is an algebra $B \subseteq A$ such that for each $\gamma \leq \vartheta$ and each $v \subseteq d(\gamma)$ there is an $f \in B$ such that $f(\gamma) = v$.

Proof. Enumerate all ordered pairs $\langle \gamma, v \rangle$, where $\gamma \leq \vartheta$ and $v \subseteq d(\gamma)$, by ordinal numbers (elements of Ω). We construct a sequence $\{B_\alpha;\ \alpha \in \Omega\}$ of countable algebras such that, for each $\alpha \in \Omega$, $B_\alpha \subseteq A$ is the algebra guaranteed by the previous theorem for the algebra $B = \bigcup \{B_\beta;\ \beta \in \alpha \cap \Omega\}$ and for the α-th pair $\langle \gamma, v \rangle$. Finally, put $B = \bigcup \{B_\alpha;\ \alpha \in \Omega\}$.

Proof of the main theorem. Let B be as in the preceding theorem. If $f \in B$ and $\gamma \leq \vartheta$ is such that $f(\gamma)$ is a one-element set, then $f(\alpha)$ is a one-element set for each $\alpha \leq \vartheta$ since f is a pointwise motion of a set. For each f which is a motion of a point in the time ϑ, put $\bar{f}(\alpha) = \{f(\alpha)\}$ for each $\alpha \leq \vartheta$. Now put $T = \{f;\ \bar{f} \in B\}$. Then T is a sheaf of motions of points in the time ϑ , since for f, $g \in T$, $f \neq g$ and $\gamma \leq \vartheta$ we cannot have $f(\gamma) = g(\gamma)$; if we had $f(\gamma) = g(\gamma)$ then we would have $\bar{f}(\gamma) = \bar{g}(\gamma)$ and taking $h \in B$ such that $h(\alpha) = \bar{f}(\alpha) - \bar{g}(\alpha)$ for each α we would have $h(\gamma) = \emptyset$, thus $h(\alpha) = \emptyset$ for each α . Furthermore, one easily sees that T is a strong decomposition of d.

Chapter V

Similarities

In classical geometry, two patterns are congruent if they differ only by their positions but cannot be otherwise distinguished. The present chapter is devoted to an analogous notion concerning classes of the extended universe. Thus, as in Chapter I, we shall investigate the extended universe, but we shall make use of notions introduced in Chapters II and III.

Our subject is not accidental: when developing mathematics in the alternative set theory we model diverse situations, capable of mathematical treatment, in the extended universe. Similarities are mappings which enable us to deal with models that are technically easily graspable because they are well located in the extended universe. Furthermore, using similarities we can investigate more deeply the structure of the extended universe.

The languages FL and FL_C play an important role throughout the chapter. Our subject forces us to use such languages since only by including formulas of various kinds in the universe of objects to be investigated can we achieve satisfactory generality. This is a minimal fragment of mathematical logic indispensable for general mathematics.

In Chapter II Section 5 we proved that the class of all set--theoretically definable classes is codable. The reader familiar with logic certainly realized that in the proof of that theorem we constructed - mutatis mutandis - the satisfaction relation for set--formulas of FL_V with respect to V. Thus in formalizing our theory we shall replace expressions of the form $\{x;\ \varphi(x)\}$, where $\varphi(x_o)$ is a set-formula of FL_V by $\{x;\ x \text{ satisfies } \varphi \text{ in } V\}$.

In Section 1 of the present chapter we shall use more general formulations - e.g. of the form "the class $\{x,\ \varphi(x,y_1,\ldots,y_n,Y_1,\ldots,Y_m)\}$ where φ is a formula of FL, $y_1, \ldots, y_n$ are sets, and $Y_1, \ldots, Y_m$ are classes from the extended universe". Such a class evidently exists for each actually realizable formula φ of L and each $y_1, \ldots, y_n, Y_1, \ldots, Y_m$, and the class of all formulas of FL is a subclass of the class of all actually realizable formulas of L.

This way of reasoning is impossible if our theory is formalized in the predicate calculus with primitive predicates $\in$ and $=$ and with variables for classes from the extended universe. In this case we have to confine ourselves to obvious modifications of the respective theorems to theorem schemata (each theorem is converted into infinitely many theorems, one for each actually realizable formula). Another possibility consists in introducing a new primitive operation Sat, associating with each formula φ of FL, each sequence $\bar{y} = \{y_n;\ n \in FN\}$ of sets, and each sequence $\bar{Y} = \{Y_n;\ n \in FN\}$ of classes a class Sat $(\varphi, \bar{y}, \bar{Y})$, and in formulating obvious axioms enabling us to replace expressions of the form
$\{x;\ \varphi(x, y_1, \ldots, y_n, Y_1, \ldots, Y_m)\}$ by Sat $(\varphi, \bar{y}, \bar{Y})$.

Section 1

Automorphisms

A function F is a <u>similarity</u> iff for each set-formula $\varphi(x_1, \ldots, x_n)$ of the language FL and for each $y_1, \ldots, y_n \in \mathrm{dom}(F)$ we have $\varphi(y_1, \ldots, y_n) \equiv \varphi(F(y_1), \ldots, F(y_n))$.

Classes X, Y are <u>similar</u> (notation: $X \approx Y$) iff there is a similarity F such that $\mathrm{dom}(F) = X$ and $\mathrm{rng}(F) = Y$.

<u>Theorem</u>. (1) Each similarity is a one-one mapping. (2) If F is a similarity then F^{-1} is also a similarity. (3) Composition of similarities is a similarity. (4) If F is a similarity and $G \subseteq F$ then G is a similarity. (5) The empty class is a similarity. (6) If $\mathcal{H}$ is a codable directed class of similarities then $\bigcup \mathcal{H}$ is a similarity.

Let φ be a formula of FL_V and let A be a constant denoting a class. Then φ^A is the formula resulting from φ by the restriction of all quantifiers binding set variables to elements of A and all quantifiers binding class variables to subclasses of A. Thus $(\forall x_i)$ and $(\exists x_i)$ are replaced by $(\forall x_i \in A)$ and $(\exists x_i \in A)$, respectively and $(\forall X_i)$ and $(\exists X_i)$ are replaced by $(\forall X_i \subseteq A)$ and $(\exists X_i \subseteq A)$, respectively.

The following theorem shows that two similar classes are indistinguishable by formulas of FL. It can be easily proved by induction on the complexity of formulas.

<u>Theorem</u>. Let $\varphi(x_1, \ldots, x_n, X_1, \ldots, X_m)$ be a formula of FL. Let F be a similarity and put $A = \mathrm{dom}(F)$, $B = \mathrm{rng}(F)$. Then

$$\varphi^A(y_1, \ldots, y_n, Y_1, \ldots, Y_m) \equiv \varphi^B(F(y_1), \ldots, F(y_n), F''Y_1, \ldots, F''Y_m)$$

for each $y_1, \ldots, y_n \in A$, $Y_1, \ldots, Y_m \subseteq A$.

In particular, it follows from the definition of a similarity and from the previous theorem that all properties characterizing the class Ω are invariant under similarities. Thus if Z is such that $Z \approx \Omega$ then Z can be identified with Ω ; such a Z may be chosen to have some useful additional properties. Similar considerations apply to other cases.

<u>Theorem</u>. Let F be a similarity and assume F to be at most countable. Then for each set u there is a v such that $F \cup \{\langle v,u\rangle\}$ is a similarity.

<u>Proof</u>. Let $\mathcal{H}$ be the class of all classes $\{x;\ \varphi(x, F(y_1), \ldots, F(y_n))\}$ where $\varphi(x_0,x_1,\ldots, x_n)$ is a set formula of FL and $y_1, \ldots, y_n$ are elements of $dom(F)$ such that $\varphi(u, y_1,\ldots,y_n)$. Evidently, $\mathcal{H}$ is codable and at most countable. Since $\{x;\ \varphi_1(x, F(y_1),\ldots, F(y_n))\} \cap \{x;\ \varphi_2(x, F(y_1),\ldots,F(y_n))\} = \{x;\ \varphi_1(x, F(y_1), \ldots, F(y_n))\ \&\ \varphi_2(x, F(y_1), \ldots, F(y_n))\}$ the clas $\mathcal{H}$ is dually directed by inclusion. From $\varphi(u, y_1,\ldots, y_n)$ we have $(\exists x_0)\varphi(x_0, y_1,\ldots, y_n)$ and consequently $(\exists x_0)\varphi(x_0,F(y_1),\ldots,F(y_n))$. Thus each element of $\mathcal{H}$ is a non-empty set-theoretically definable class. It follows that $\bigcap \mathcal{H} \neq \emptyset$. Take a $v \in \bigcap \mathcal{H}$. Then $F \cup \{\langle v,u\rangle\}$ is a similarity.

A similarity whose domain and range is V is called an <u>automorphism</u>.

<u>Theorem</u>. Let F be an automorphism. Then for each set u we have $F(u) = F''u$.

<u>Proof</u>. Evidently, $x \in u$ implies $F(x) \in F(u)$, thus $F''u \subseteq F(u)$. If $y \in F(u)$ then $y = F(x)$ for some x and $F(x) \in F(u)$. It follows $x \in u$, thus $y \in F''u$. We have proved $F(u) \subseteq F''u$.

<u>Corollary</u>. If F is an automorphism then X is a set iff $F''X$ is a set.

<u>Theorem</u>. Let $\varphi(x_1,\ldots,x_n, X_1,\ldots,X_m)$ be a formula in FL. Let F be an automorphism. Then

$$\varphi(y_1,\ldots,y_n,Y_1,\ldots,Y_m) \equiv \varphi(F(y_1),\ldots,F(y_n), F''Y_1,\ldots,F''Y_m)$$

for each $y_1,\ldots,y_n, Y_1,\ldots,Y_m$.

<u>Theorem</u>. If F_0 is a similarity, F_0 at most countable, then there is an automorphism F such that $F_0 \subseteq F$.

<u>Proof</u>. Let $\{x_\alpha;\ 0 \neq \alpha \in \Omega\}$ be an enumeration of all elements of V. We construct a class $\{F_\alpha;\ \alpha \in \Omega\}$ of similarities inductively as follows: for each non-zero $\alpha \in \Omega$, F_α is a similarity which is at most countable such that $\bigcup\{F_\beta;\ \beta \in \Omega \cap \alpha\} \subseteq F_\alpha$, $x_\alpha \in dom(F_\alpha)$,

and $x_\alpha \in rng(F_\alpha)$. Existence of such an F_α follows from preceding theorems; to be definite, take for F_α the first mapping having all desired properties in a well-ordering of the class $P_\omega(V)$. Finally, put $F = \bigcup \{F_\alpha ; \alpha \in \Omega\}$, evidently, F is an automorphism and $F_0 \subseteq F$.

- x - X - x -

A set y is <u>definable</u> iff there is a set-formula $\varphi(x_0)$ of FL such that y is the only set satisfying φ, i.e. $\varphi(y)$ & $(\exists! x_0)\varphi(x_0)$. The class of all definable sets is denoted by Def.

<u>Theorem</u>. For each y, y is definable iff $F(y) = y$ for each automorphism F.

<u>Proof</u>. Let y be definable by $\varphi(x_0)$ and let F be an automorphism. Then $\varphi(F(y))$ and consequently $F(y) = y$. On the other hand, assume that y is not definable. Let $\mathcal{H}$ be the class consisting of all classes $\{x; \varphi(x)\}$, where φ is a set-formula in FL and y satisfies φ. Then $\mathcal{H}$ is codable, non-emty, at most countable, and $y \in \bigcap \mathcal{H}$. Assume $\bigcap \mathcal{H} = \{y\}$; then $\{y\}$ can be obtained as the intersection of finitely many elements of $\mathcal{H}$, since $\mathcal{H}$ consists of set-theoretically definable classes. Consequently, there are set formulas $\varphi_1,\dots,\varphi_n$ in FL such that $\varphi_1(y)$ & ... & $\varphi_n(y)$ & $(\exists! x_0)(\varphi_1(x_0)$ & ... & $\varphi_n(x_0))$. Thus y is definable, which is a contradiction. Thus there is a $z \in \bigcap \mathcal{H}$ such that $z \neq y$. Since z satisfies the same set-formulas of FL, $\{\langle z,y\rangle\}$ is a similarity. There is an automorphism F such that $F(y) = z$, thus $F(y) \neq y$.

<u>Theorem</u>. A set y is definable iff there is an arbitrary formula $\varphi(x_0)$ of FL such that $\varphi(y)$ and $(\exists! x_0)\varphi(x_0)$.

<u>Proof</u>. By definition, if y is definable then there is a set-formula $\varphi(x_0)$ of FL such that $\varphi(y)$ & $(\exists! x_0)\varphi(x_0)$. On the other hand, if $\varphi(x_0)$ is an arbitrary formula of FL and F is an automorphism, then $\varphi(y)$ implies $\varphi(F(y))$. Thus if y is the only set satisfying φ then $F(y) = y$. Hence y is a fixed-point of all automorphisms and consequently y is definable.

<u>Theorem</u>. If $y_1,\dots,y_n$ are definable, $\varphi(x_0,x_1,\dots,x_n)$ is a formula of FL, and $y_{n+1} = \{x; \varphi(x,y_1,\dots,y_n)\}$, then y_{n+1} is definable.

<u>Proof</u>. Let F be an automorphism. Then $F(y_{n+1}) = F''y_{n+1}$. Thus it

suffices to prove that $y_{n+1} = F''y_{n+1}$. Since $\varphi(x,y_1,\dots,y_n) \equiv \varphi(F(x), F(y_1), \dots, F(y_n))$ for each x, we have $\varphi(x,y_1,\dots,y_n) \equiv \varphi(F(x), y_1,\dots,y_n)$ for each x. Let z be an arbitrary element of y_{n+1} and let x be such that $z = F(x)$. We have $\varphi(z,y_1,\dots,y_n)$, which implies $\varphi(x,y_1,\dots,y_n)$, hence $x \in y_{n+1}$ and $z \in F''y_{n+1}$. We have proved $y_{n+1} \subseteq F''y_{n+1}$; the converse implication is proved analogously.

Corollary. If $y_1,\dots,y_n \in \mathrm{Def}$ then $\{y_1,\dots,y_n\}$, $y_1 \cap y_2$, $y_1 \cup y_2$, $P(y_1)$, $\bigcup y_1$ etc. are definable sets.

Evidently, $\emptyset$ is definable. One can prove easily by induction that each finite natural number is also definable.

Theorem. The class Def is countable.

Proof. Since there are countably many set-formulas in FL, Def is at most countable. Since $FN \subseteq \mathrm{Def}$, Def is not finite.

Theorem. Each subset of Def is an element of Def.

Proof. Since Def is countable, each subset u of Def is finite. Hence $u = \{x_1,\dots,x_n\}$ for some $x_1,\dots,x_n \in \mathrm{Def}$.

Theorem. If $\varphi(x_0)$ is a set-formula in FL and the class $X = \{x;\ \varphi(x)\}$ is non-empty, then X has a definable element.

Proof. In Chapter II Section 1 we proved that there is a set-theoretically definable one-one mapping of V on N. An inspection of our proof shows that the constructed mapping can be described by a set-formula in FL (i.e. without parameters). Thus we can construct a set-formula $\vartheta(x_1,x_2)$ in FL such that the class $R = \{\langle x,y\rangle;\ \vartheta(x,y)\}$ is a linear ordering of V such that for each x the initial segment determined by x is a set. Let $\psi(x_0)$ be the formula $\varphi(x_0)\ \&\ (\forall x_1)(\varphi(x_1) \Rightarrow \vartheta(x_0,x_1))$. Thus $\psi(x_0)$ says that x_0 is the first element satisfying φ (in the ordering R). Evidently, there is exactly one u satisfying ψ; this u is a definable element of X.

Theorem. If u is definable and non-empty then u has a definable element.

Proof. Let $\psi(x_0)$ be a set-formula in FL defining u and let $\varphi(x_0)$ be the formula $(\exists x_1)(\psi(x_1)\ \&\ x_0 \in x_1)$; then $u = \{x;\ \varphi(x)\}$. By the preceding theorem, u has a definable element.

- x - X - x -

Define $x \overset{\circ}{=} y$ iff for each set-formula $\varphi(x_0)$ in FL we have $\varphi(x) \equiv \varphi(y)$. Evidently, $\overset{\circ}{=}$ is an equivalence.

<u>Theorem</u>. $\overset{\circ}{=}$ is an indiscernibility equivalence.

<u>Proof.</u> For each set-formula $\varphi(x_0)$ in FL put $R_\varphi = \{\langle x,y\rangle ;\ \varphi(x) \equiv \varphi(y)\}$. Evidently, R_φ is a set-theoretically definable equivalence having only two factors, $\{x;\ \varphi(x)\}$ and $\{x;\ \neg\varphi(x)\}$. Thus R_φ is an indiscernibility. Now, $\overset{\circ}{=}$ is the intersection of all equivalences R_φ, hence the intersection of countably many indiscernibility equivalences and consequently, $\overset{\circ}{=}$ is an indiscernibility equivalence.

<u>Theorem</u>. $\overset{\circ}{=}$ is totally disconnected.

<u>Proof</u>. By forming succesive finite intersection of the equivalences R_φ we can easily obtain a generating sequence for $\overset{\circ}{=}$ consisting of equivalences.

The following theorem follows immediately from the above results.

<u>Theorem</u>. (1) If F is a similarity and $x \in \mathrm{dom}(F)$, then $x \overset{\circ}{=} F(x)$.
(2) $x \overset{\circ}{=} y$ iff there is an automorphism F such that $y = F(x)$.
(3) $x \overset{\circ}{=} y$ iff for each formula $\varphi(x_0)$ of FL we have $\varphi(x) \equiv \varphi(y)$.

In the sequal all notions introduced in Chapter III are used with respect to the equivalence $\overset{\circ}{=}$.

<u>Theorem</u>. For each formula $\varphi(x_0)$ of FL, the class $\{x;\ \varphi(x)\}$ is a figure.

<u>Proof</u>. Put $X = \{x;\ \varphi(x)\}$. Let $y \in X$ and $z \overset{\circ}{=} y$. Let F be an automorphism such that $z = F(y)$. Now, $\varphi(y)$ implies $\varphi(F(y))$, hence $\varphi(z)$. Thus $z \in X$.

<u>Theorem</u>. Let $\varphi(X_0)$ be a formula of FL such that $(\exists! X_0)\varphi(X_0)$. Then the unique Y satisfying φ is a figure.

<u>Proof</u>. Let $\psi(x_0)$ be the formula $(\exists X_0)(\varphi(X_0)\ \&\ x_0 \in X_0)$. Then $Y = \{x;\ \psi(x)\}$ and Y is a figure by the preceding theorem.

<u>Theorem</u>. For each Y, Y is a clopen figure iff there is a set-formula of FL such that $Y = \{x;\ \varphi(x)\}$.

<u>Proof</u>. The implication $\Leftarrow$ is immediate (Y is a set-theoretically

definable figure). Conversely, assume that Y is a figure and Y is set-theoretically definable. Let $\mathcal{X}$ be the class consisting of all classes $\{x;\ \varphi(x)\}$ where $\varphi(x_0)$ is a set-formula of FL and $Y \subseteq \{x;\ \varphi(x)\}$. Then $\mathcal{X}$ is codable, at most countable, dually directed by inclusion, and $Y \subseteq \bigcap \mathcal{X}$. We claim $Y = \bigcap \mathcal{X}$. Let $y \notin Y$. Let $\mathcal{M}$ be the class consisting of all classes $\{x;\ \varphi(x)\}$ where $\varphi(x_0)$ is a set-formula of FL satisfied by y. $\mathcal{M}$ is codable, at most countable, and dually directed by inclusion. If we had $X \cap Y \neq \emptyset$ for each $X \in \mathcal{M}$ then we would have $\bigcap \mathcal{M} \cap Y \neq \emptyset$, thus $\mathrm{Mon}(y) \cap Y \neq \emptyset$ and $y \in Y$ since Y is a figure. Hence there is a set-formula $\varphi(x_0)$ of FL satisfied by y and such that $\{x;\ \varphi(x)\} \cap Y = \emptyset$. This implies $\{x;\ \neg\varphi(x)\} \in \mathcal{X}$ and $y \notin \bigcap \mathcal{X}$. We have proved our claim. Since Y is set-theoretically definable, $\mathcal{X}$ is at most countable and dually directed, and $\mathcal{X}$ contains only set-theoretically definable classes, we have $Y \in \mathcal{X}$. Thus there is a set-formula $\varphi(x_0)$ of FL (i.e. without constants!) such that $Y = \{x;\ \varphi(x)\}$.

<u>Theorem</u>. The class Def is dense in V.

<u>Proof</u>. The class $\overline{\mathrm{Def}}$ (closure of Def) is closed, hence a π-clas If $y \notin \overline{\mathrm{Def}}$ then $\mathrm{Mon}(y) \cap \overline{\mathrm{Def}} = \emptyset$. Let $\mathcal{X}$ be the class consisting of all classes $\{x;\ \varphi(x)\}$ where $\varphi(x)$ is a set-formula of FL satisfied by y. $\mathcal{X}$ is codable, at most countable, dually directed, and $\mathrm{Mon}(y) = \bigcap \mathcal{X}$. If we had $X \cap \overline{\mathrm{Def}} \neq \emptyset$ for each $X \in \mathcal{X}$ then we would have $\bigcap \mathcal{X} \cap \overline{\mathrm{Def}} \neq \emptyset$, hence $\mathrm{Mon}(y) \cap \overline{\mathrm{Def}} \neq \emptyset$. Hence there exists a set-formul $\varphi(x_0)$ of FL satisfied by y and such that $\overline{\mathrm{Def}} \cap \{x;\ \varphi(x)\} = \emptyset$. But by theorem above, $\{x;\ \varphi(x)\}$ must have a definable element - a contra diction.

<u>Theorem</u>. (1) If $y \in \mathrm{Def}$ then $\mathrm{Mon}(y) = \{y\}$. (2) If Mon(y) is a set-theoretically definable class then $y \in \mathrm{Def}$.

<u>Proof.</u> (1) is trivial. (2) Assume that Mon(y) is clopen. Then there is a set-formula $\varphi(x_0)$ of FL such that $\mathrm{Mon}(y) = \{x;\ \varphi(x)\}$. But $\{x;\ \varphi(x)\}$ has a definable element, i.e. there is a $z \in \mathrm{Def}$ such that $y \overset{\circ}{=} z$. But (1), $\mathrm{Mon}(z) = \{z\}$, hence $y = z$ and $y \in \mathrm{Def}$.

<u>Corollary</u>. For each $y \notin \mathrm{Def}$, Mon(y) is an uncountable π-class.

<u>Theorem</u>. Let $\varphi(X_0)$ be a formula of FL such that $(\exists! X_0)\ \varphi(X_0)$. If the unique Y satisfying φ is countable then $Y \subseteq \mathrm{Def}$.

<u>Proof</u>. Assume $y \in Y - \mathrm{Def}$. Then $\mathrm{Mon}(y) \subseteq Y$, since Y is a figure. But Mon(y) is uncountable, a contradiction.

We can easily construct a formula $\varphi(X_0)$ of FL such that $(\exists! X_0)\varphi(X_0)$ and such that the unique class satisfying φ is Def. Thus Def is the largest countable class definable in this way.

<u>Theorem</u>. Let X, Y be clopen figures. Then $X \cap \mathrm{Def} = Y \cap \mathrm{Def}$ iff $X = Y$.

<u>Proof</u>. Let $\varphi(x_0)$, $\psi(x_0)$ be set-formulas of FL defining X and Y respectively. If, for example, y satisfies φ but not ψ, then $\{x;\ \varphi(x)\ \&\ \neg\psi(x)\}$ is non-empty and therefore has a definable element.

The assertion that Def has an uncountable element is independent of the axioms we have assumed.

Section 2

Endomorphisms

Throughout this section, $\mathcal{M}$ denotes an ultrafilter on the ring $\{X;\ Sd(X)\}$ of all set-theoretically definable classes.

F, $\mathcal{M}$ and d are <u>coherent</u> if
$\{x;\ \varphi(x,y_1,\dots,y_n)\} \in \mathcal{M} \equiv \varphi(d,\ F(y_1),\ \dots,\ F(y_n))$
for each set-formula φ of FL and each $y_1,\dots,y_n \in dom(F)$.

<u>Theorem</u>. (1) If F, $\mathcal{M}$, d are coherent then F is a similarity.
(2) If F, $\mathcal{M}$, d are coherent and $G \subseteq F$, then G, $\mathcal{M}$, d are coherent.
(3) Let $\mathcal{H}$ be a codable non-empty class directed by inclusion and such that each element of $\mathcal{H}$ is a similarity and $F \in \mathcal{H}$ implies that F, $\mathcal{M}$, d are coherent. Then $\bigcup \mathcal{H}$, $\mathcal{M}$, d are coherent.

<u>Theorem</u>. If F is a similarity which is at most countable, then there is a d such that F, $\mathcal{M}$, d are coherent.

<u>Proof</u>. Let $\mathcal{H}$ consist of all classes $\{z;\ \varphi(z,F(y_1),\dots,F(y_n))\}$ where (1) φ is a set-formula of FL, (2) $y_1,\dots,y_n \in dom(F)$ and (3) $\{x;\ \varphi(x,y_1,\dots,y_n)\} \in \mathcal{M}$. Then $\mathcal{H}$ is codable, at most countable dually directed, and each element of $\mathcal{H}$ is non-empty set-theoreticall definable class. It follows that $\bigcap \mathcal{H} \neq \emptyset$. Pick a $d \in \bigcap \mathcal{H}$; then F, $\mathcal{M}$, d are coherent.

<u>Theorem</u>. If F, $\mathcal{M}$, d are coherent and F is at most countable, then for each set u there is a v such that $F \cup \{\langle v,u\rangle\}$, $\mathcal{M}$, d are coherent.

<u>Proof</u>. Let $\mathcal{H}$ consist of all classes $\{z;\ \varphi(d,z,F(y_2),\dots,F(y_n))\}$ where (1) φ is a set-formula of FL, (2) $y_2,\dots,y_n \in dom(F)$ and (3) $\{y;\ \varphi(y,u,y_2,\dots,y_n)\} \in \mathcal{M}$. Then $\mathcal{H}$ is codable, at most countable, dually directed and each element of $\mathcal{H}$ is set-theoretically definable. Assume $\{z;\ \varphi(d,z,\ F(y_2),\ \dots,\ F(y_n)\} \in \mathcal{H}$. Then
$\{y;\ \varphi(y,u,y_2,\dots,y_n)\} \in \mathcal{M}$ and consequently
$\{y;\ (\exists x_1)\ \varphi(y,x_1,y_2,\dots,y_n)\} \in \mathcal{M}$. Hence each element of $\mathcal{H}$ is non-empty so $\bigcap \mathcal{H} \neq \emptyset$. Pick a $v \in \bigcap \mathcal{H}$; then $F \cup \{\langle v,u\rangle\}$, $\mathcal{M}$, d are coherent.

A similarity whose domain is V is called an <u>endomorphism</u>.

Theorem. Let F_0, $\mathfrak{M}$, d be coherent and let F_0 be at most countable. Then there is an endomorphism F such that $F_0 \subseteq F$ and F, $\mathfrak{M}$, d are coherent.

Proof. Let $\{x_\alpha ; 0 \neq \alpha \in \Omega\}$ be an enumeration of V. We shall construct a sequence $\{F_\alpha ; \alpha \in \Omega\}$ such that F_α, $\mathfrak{M}$, d are coherent for each $\alpha \in \Omega$, each F_α is at most countable, $x_\alpha \in \mathrm{dom}(F_\alpha)$, and $\bigcup\{F_\beta ; \beta \in \alpha \cap \Omega\} \subseteq F_\alpha$. For each α, the existence of such an F_α is guaranteed by the above theorems; thus we simply pick the first possible candidate in a fixed well-ordering of $P_\omega(V)$. Finally, we put $F = \bigcup\{F_\alpha ; \alpha \in \Omega\}$. Evidently, F is an endomorphism, $F_0 \subseteq F$, and F, $\mathfrak{M}$, d are coherent.

The preceding theorem has several important consequences; we are going to present some of them.

Theorem. There is a semiset similar to the universal class V.

Proof. Let $\mathfrak{M}$ be an ultrafilter such that, for each y, $\{x; y \in x\} \in \mathfrak{M}$. (Evidently, there is such an $\mathfrak{M}$.) Let F be an endomorphism and let d be a set such that F, $\mathfrak{M}$, d are coherent. Put $A = F"V$. Then $A \approx V$. Since $\{x; y \in x\} \in \mathfrak{M}$ for each y, we have $F(y) \in d$ and consequently $A \subseteq d$.

This theorem makes precise of our vague remarks to the effect that in all considerations in the alternative set theory V can be replaced by an appropriate semiset.

Theorem. For each X there is a Y and a set d such that $X \approx Y$, $Y \subseteq d$ and

$$((\forall x)(x \subseteq X \ \&\ \mathrm{Fin}(x) \Rightarrow \varphi(x))) \Rightarrow \varphi(d)$$

for each set-formula φ of FL.

Proof. Let $\mathfrak{M}$ be such that $\{x; y \in x\} \in \mathfrak{M}$ for each $y \in X$ and, moreover, $\{x; \varphi(x)\} \in \mathfrak{M}$ for each set-formula φ of FL satisfied by each finite subset of X. Evidently, there is such an $\mathfrak{M}$. Let F be an endomorphism and d a set such that F, $\mathfrak{M}$, d are coherent. Put $Y = F"X$. If $y \in X$ then $\{x; y \in x\} \in \mathfrak{M}$ and hence $F(y) \in d$; thus $Y \subseteq d$. If $\varphi(x_0)$ is a set-formula of FL satisfied by each finite subset of X then $\{x; \varphi(x)\} \in \mathfrak{M}$ and consequently $\varphi(d)$.

Observe that each class similar to a function is itself a function and that each subset of a function is a function. Thus we have the following.

Corollary. For each function F there are functions G and g su‹
that $F \approx G$, $G \subseteq g$ and, in addition,

$$(\forall f \subseteq F)(Fin(f) \Rightarrow \varphi(f)) \Rightarrow \varphi(g)$$

for each set-formula φ of FL.

This is a strengthening of the axiom of prolongation since by this corollary, also various uncountable functions have prolongati‹ Trivially, not every function can be prolonged to a function which a set; but for each function there is a similar function having a prolongation to a function which is a set.

All above notions and theorems involving formulas of FL can b‹ generalized by replacing FL by FL_C where C is at most countable class. Details are left to the reader as an exercise.

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